19. (a) Using $A = A_0\left(1+\frac{r}{n}\right)^{nt}$ with $A_0 = 3000$, $r = 0.05$, and $t = 5$, we have:

(i) Annually: $n = 1$; $A = 3000\left(1+\frac{0.05}{1}\right)^{1\cdot 5} = \$3828.84$

(ii) Semiannually: $n = 2$; $A = 3000\left(1+\frac{0.05}{2}\right)^{2\cdot 5} = \$3840.25$

(iii) Monthly: $n = 12$; $A = 3000\left(1+\frac{0.05}{12}\right)^{12\cdot 5} = \$3850.08$

(iv) Weekly: $n = 52$; $A = 3000\left(1+\frac{0.05}{52}\right)^{52\cdot 5} = \$3851.61$

(v) Daily: $n = 365$; $A = 3000\left(1+\frac{0.05}{365}\right)^{365\cdot 5} = \$3852.01$

(vi) Continuously: $A = 3000e^{(0.05)5} = \$3852.08$

(b) $dA/dt = 0.05A$ and $A(0) = 3000$.

3.9 Related Rates

1. $V = x^3 \;\Rightarrow\; \dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt} = 3x^2\dfrac{dx}{dt}$

3. Let s denote the side of a square. The square's area A is given by $A = s^2$. Differentiating with respect to t gives us $\dfrac{dA}{dt} = 2s\dfrac{ds}{dt}$. When $A = 16$, $s = 4$. Substitution 4 for s and 6 for $\dfrac{ds}{dt}$ gives us $\dfrac{dA}{dt} = 2(4)(6) = 48\text{ cm}^2/\text{s}$.

5. $V = \pi r^2 h = \pi(5)^2 h = 25\pi h \;\Rightarrow\; \dfrac{dV}{dt} = 25\pi\dfrac{dh}{dt} \;\Rightarrow\; 3 = 25\pi\dfrac{dh}{dt} \;\Rightarrow\; \dfrac{dh}{dt} = \dfrac{3}{25\pi}$ m/min.

7. (a) $y = \sqrt{2x+1}$ and $\dfrac{dx}{dt} = 3 \;\Rightarrow\; \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{1}{2}(2x+1)^{-1/2}\cdot 2\cdot 3 = \dfrac{3}{\sqrt{2x+1}}$. When $x = 4$, $\dfrac{dy}{dt} = \dfrac{3}{\sqrt{9}} = 1$.

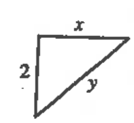
(b) $y = \sqrt{2x+1} \;\Rightarrow\; y^2 = 2x+1 \;\Rightarrow\; 2x = y^2 - 1 \;\Rightarrow\; x = \frac{1}{2}y^2 - \frac{1}{2}$ and $\dfrac{dy}{dt} = 5 \;\Rightarrow$ $\dfrac{dx}{dt} = \dfrac{dx}{dy}\dfrac{dy}{dt} = y\cdot 5 = 5y$. When $x = 12$, $y = \sqrt{25} = 5$, so $\dfrac{dx}{dt} = 5(5) = 25$.

9. $\dfrac{d}{dt}(x^2+y^2+z^2) = \dfrac{d}{dt}(9) \;\Rightarrow\; 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} + 2z\dfrac{dz}{dt} = 0 \;\Rightarrow\; x\dfrac{dx}{dt} + y\dfrac{dy}{dt} + z\dfrac{dz}{dt} = 0$. If $\dfrac{dx}{dt} = 5$, $\dfrac{dy}{dt} = 4$ and $(x,y,z) = (2,2,1)$, then $2(5) + 2(4) + 1\dfrac{dz}{dt} = 0 \;\Rightarrow\; \dfrac{dz}{dt} = -18$.

11. (a) Given: a plane flying horizontally at an altitude of 2 km and a speed of 800 km/h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in km), then we are given that $dx/dt = 800$ km/h.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 3 km from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 3$ km.

(c)

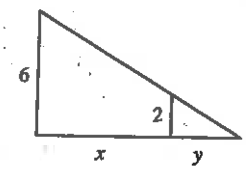


(d) By the Pythagorean Theorem, $y^2 = x^2 + 2^2 \;\Rightarrow\; 2y\,(dy/dt) = 2x\,(dx/dt)$.

(e) $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{x}{y}(800)$. Since $y^2 = x^2 + 4$, when $y = 3$, $x = \sqrt{5}$, so $\dfrac{dy}{dt} = \frac{\sqrt{5}}{3}(800) \approx 596$ km/h.

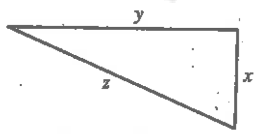
13. (a) Given: a man 2 m tall walks away from a street light mounted on a 6-m-tall pole at a rate of 1.5 m/s. If we let t be time (in seconds) and x be the distance from the pole to the man (in meters), then we are given that $dx/dt = 1.5$ m/s.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 10 m from the pole. If we let y be the distance from the man to the tip of his shadow (in meters), then we want to find $\frac{d}{dt}(x+y)$ when $x = 10$ m.

(c) 

(d) By similar triangles, $\frac{6}{2} = \frac{x+y}{y} \Rightarrow 6y = 2x + 2y \Rightarrow 4y = 2x \Rightarrow y = \frac{1}{2}x$.

(e) The tip of the shadow moves at a rate of $\frac{d}{dt}(x+y) = \frac{d}{dt}\left(x + \frac{1}{2}x\right) = \frac{3}{2}\frac{dx}{dt} = \frac{3}{2}(1.5) = \frac{9}{4}$ m/s.

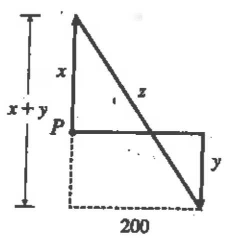
15. 

We are given that $\frac{dx}{dt} = 30$ km/h and $\frac{dy}{dt} = 72$ km/h. $z^2 = x^2 + y^2 \Rightarrow$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$.

After 2 hours, $x = 2\,(30) = 60$ and $y = 2\,(72) = 144 \Rightarrow z = \sqrt{60^2 + 144^2} = 156$, so

$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{60(30) + 144(72)}{156} = 78$ km/h.

17. 

We are given that $\frac{dx}{dt} = 1.2$ m/s and $\frac{dy}{dt} = 1.6$ m/s. $z^2 = (x+y)^2 + 200^2 \Rightarrow$

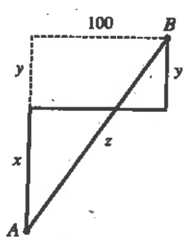
$2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$. 15 minutes after the woman starts, we have

$x = (1.2 \text{ m/s})(20 \text{ min})(60 \text{ s/min}) = 1440$ m and $y = 1.6 \cdot 15 \cdot 60 = 1440 \Rightarrow$

$z = \sqrt{(1440 + 1440)^2 + 200^2} = \sqrt{8\,334\,400}$, so

$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{1440 + 1440}{\sqrt{8\,334\,400}}(1.2 + 1.6) = \frac{8064}{\sqrt{8\,334\,400}} \approx 2.79$ m/s.

19. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\frac{dh}{dt} = 1$ cm/min and $\frac{dA}{dt} = 2$ cm^2/min. Using the Product Rule, we have $\frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have $100 = \frac{1}{2}b(10) \Rightarrow \frac{1}{2}b = 10 \Rightarrow$ $b = 20$, so $2 = \frac{1}{2}\left(20 \cdot 1 + 10\frac{db}{dt}\right) \Rightarrow 4 = 20 + 10\frac{db}{dt} \Rightarrow \frac{db}{dt} = \frac{4-20}{10} = -1.6$ cm/min.

21. 

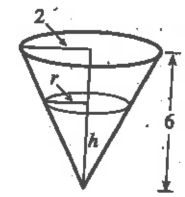
We are given that $\frac{dx}{dt} = 35$ km/h and $\frac{dy}{dt} = 25$ km/h. $z^2 = (x+y)^2 + 100^2 \Rightarrow$

$2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$. At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \Rightarrow$

$z = \sqrt{(140 + 100)^2 + 100^2} = \sqrt{67{,}600} = 260$, so

$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{140 + 100}{260}(35 + 25) = \frac{720}{13} \approx 55.4$ km/h.

23. If $C =$ the rate at which water is pumped in, then $\frac{dV}{dt} = C - 10{,}000$, where

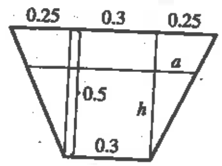


$V = \frac{1}{3}\pi r^2 h$ is the volume at time t. By similar triangles, $\frac{r}{2} = \frac{h}{6} \Rightarrow r = \frac{1}{3}h \Rightarrow$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$. When $h = 200$ cm,

$\frac{dh}{dt} = 20$ cm/min, so $C - 10{,}000 = \frac{\pi}{9}(200)^2(20) \Rightarrow C = 10{,}000 + \frac{800{,}000}{9}\pi \approx 289{,}253 \text{ cm}^3/\text{min}$.

25.



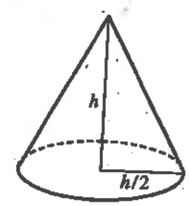
The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is $V = (10)\frac{1}{2}[0.3 + (0.3 + 2a)]h$.

By similar triangles, $\frac{a}{h} = \frac{0.25}{0.5} = \frac{1}{2}$, so $2a = h \Rightarrow V = 5(0.6 + h)h = 3h + 5h^2$.

Now $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \Rightarrow 0.2 = (3 + 10h)\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{0.2}{3 + 10h}$. When $h = 0.3$,

$\frac{dh}{dt} = \frac{0.2}{3 + 10(0.3)} = \frac{0.2}{6}$ m/min $= \frac{1}{30}$ m/min or $\frac{10}{3}$ cm/min.

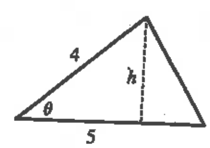
27. We are given that $\frac{dV}{dt} = 3 \text{ m}^3/\text{min}$. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12} \Rightarrow$



$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \Rightarrow 3 = \frac{\pi h^2}{4}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{12}{\pi h^2}$. When $h = 3$ m,

$\frac{dh}{dt} = \frac{12}{3^2\pi} = \frac{4}{3\pi} \approx 0.42$ m/min.

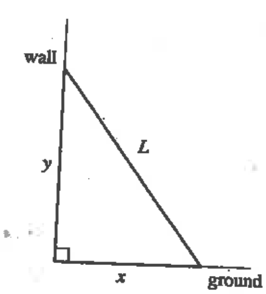
29. $A = \frac{1}{2}bh$, but $b = 5$ m and $\sin\theta = \frac{h}{4} \Rightarrow h = 4\sin\theta$, so $A = \frac{1}{2}(5)(4\sin\theta) = 10\sin\theta$.



We are given $\frac{d\theta}{dt} = 0.06$ rad/s, so $\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = (10\cos\theta)(0.06) = 0.6\cos\theta$.

When $\theta = \frac{\pi}{3}$, $\frac{dA}{dt} = 0.6\left(\cos\frac{\pi}{3}\right) = (0.6)\left(\frac{1}{2}\right) = 0.3 \text{ m}^2/\text{s}$.

31. From the figure and given information, we have $x^2 + y^2 = L^2$, $\frac{dy}{dt} = -0.15$ m/s, and $\frac{dx}{dt} = 0.2$ m/s when $x = 3$ m. Differentiating implicitly with respect to t, we get



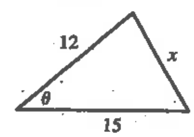
$x^2 + y^2 = L^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow y\frac{dy}{dt} = -x\frac{dx}{dt}$. Substituting the given information gives us $y(-0.15) = -3(0.2) \Rightarrow y = 4$ m. Thus, $3^2 + 4^2 = L^2 \Rightarrow L^2 = 25 \Rightarrow L = 5$ m.

33. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\dfrac{dV}{dt} + V\dfrac{dP}{dt} = 0 \Rightarrow$

$\dfrac{dV}{dt} = -\dfrac{V}{P}\dfrac{dP}{dt}$. When $V = 600$, $P = 150$ and $\dfrac{dP}{dt} = 20$, so we have $\dfrac{dV}{dt} = -\dfrac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of 80 cm^3/min.

35. With $R_1 = 80$ and $R_2 = 100$, $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{1}{80} + \dfrac{1}{100} = \dfrac{180}{8000} = \dfrac{9}{400}$, so $R = \dfrac{400}{9}$. Differentiating $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ with respect to t, we have $-\dfrac{1}{R^2}\dfrac{dR}{dt} = -\dfrac{1}{R_1^2}\dfrac{dR_1}{dt} - \dfrac{1}{R_2^2}\dfrac{dR_2}{dt} \Rightarrow \dfrac{dR}{dt} = R^2\left(\dfrac{1}{R_1^2}\dfrac{dR_1}{dt} + \dfrac{1}{R_2^2}\dfrac{dR_2}{dt}\right)$. When $R_1 = 80$ and $R_2 = 100$, $\dfrac{dR}{dt} = \dfrac{400^2}{9^2}\left[\dfrac{1}{80^2}(0.3) + \dfrac{1}{100^2}(0.2)\right] = \dfrac{107}{810} \approx 0.132\ \Omega/\text{s}$.

37. We are given $d\theta/dt = 2°/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,

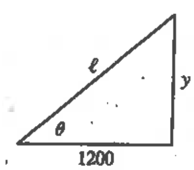


$x^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta = 369 - 360\cos\theta \Rightarrow$

$2x\dfrac{dx}{dt} = 360\sin\theta\dfrac{d\theta}{dt} \Rightarrow \dfrac{dx}{dt} = \dfrac{180\sin\theta}{x}\dfrac{d\theta}{dt}$. When $\theta = 60°$,

$x = \sqrt{369 - 360\cos 60°} = \sqrt{189} = 3\sqrt{21}$, so $\dfrac{dx}{dt} = \dfrac{180\sin 60°}{3\sqrt{21}}\dfrac{\pi}{90} = \dfrac{\pi\sqrt{3}}{3\sqrt{21}} = \dfrac{\sqrt{7}\,\pi}{21} \approx 0.396$ m/min.

39. (a) By the Pythagorean Theorem, $1200^2 + y^2 = \ell^2$. Differentiating with respect to t,



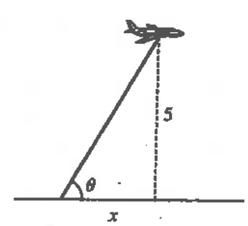
we obtain $2y\dfrac{dy}{dt} = 2\ell\dfrac{d\ell}{dt}$. We know that $\dfrac{dy}{dt} = 200$ m/s, so when $y = 900$ m,

$\ell = \sqrt{1200^2 + 900^2} = \sqrt{2\,250\,000} = 1500$ m and $\dfrac{d\ell}{dt} = \dfrac{y}{\ell}\dfrac{dy}{dt} = \dfrac{900}{1500}(200) = 120$ m/s.

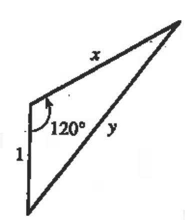
(b) Here $\tan\theta = \dfrac{y}{1200} \Rightarrow \dfrac{d}{dt}(\tan\theta) = \dfrac{d}{dt}\left(\dfrac{y}{1200}\right) \Rightarrow \sec^2\theta\dfrac{d\theta}{dt} = \dfrac{1}{1200}\dfrac{dy}{dt} \Rightarrow \dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{1200}\dfrac{dy}{dt}$. When $y = 900$ m, $\dfrac{dy}{dt} = 200$ m/s, $\ell = 1500$ and $\cos\theta = \dfrac{1200}{\ell} = \dfrac{1200}{1500} = \dfrac{4}{5}$, so $\dfrac{d\theta}{dt} = \dfrac{(4/5)^2}{1200}(200) \approx 0.107$ rad/s.

41. $\cot\theta = \dfrac{x}{5} \Rightarrow -\csc^2\theta\dfrac{d\theta}{dt} = \dfrac{1}{5}\dfrac{dx}{dt} \Rightarrow -\left(\csc\dfrac{\pi}{3}\right)^2\left(-\dfrac{\pi}{6}\right) = \dfrac{1}{5}\dfrac{dx}{dt} \Rightarrow$

$\dfrac{dx}{dt} = \dfrac{5\pi}{6}\left(\dfrac{2}{\sqrt{3}}\right)^2 = \dfrac{10}{9}\pi$ km/min [≈ 130 mi/h]



43. We are given that $\dfrac{dx}{dt} = 300$ km/h. By the Law of Cosines,



$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120° = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1$, so

$2y\dfrac{dy}{dt} = 2x\dfrac{dx}{dt} + \dfrac{dx}{dt} \Rightarrow \dfrac{dy}{dt} = \dfrac{2x+1}{2y}\dfrac{dx}{dt}$. After 1 minute, $x = \frac{300}{60} = 5$ km $\Rightarrow$

$y = \sqrt{5^2 + 5 + 1} = \sqrt{31}$ km $\Rightarrow \dfrac{dy}{dt} = \dfrac{2(5)+1}{2\sqrt{31}}(300) = \dfrac{1650}{\sqrt{31}} \approx 296$ km/h.