3.5 Implicit Differentiation

1. (a) $\frac{d}{dx}(xy + 2x + 3x^2) = \frac{d}{dx}(4) \Rightarrow (x \cdot y' + y \cdot 1) + 2 + 6x = 0 \Rightarrow xy' = -y - 2 - 6x \Rightarrow$
$y' = \frac{-y-2-6x}{x}$ or $y' = -6 - \frac{y+2}{x}$.

(b) $xy + 2x + 3x^2 = 4 \Rightarrow xy = 4 - 2x - 3x^2 \Rightarrow y = \frac{4-2x-3x^2}{x} = \frac{4}{x} - 2 - 3x$, so $y' = -\frac{4}{x^2} - 3$.

(c) From part (a), $y' = \frac{-y-2-6x}{x} = \frac{-(4/x - 2 - 3x) - 2 - 6x}{x} = \frac{-4/x - 3x}{x} = -\frac{4}{x^2} - 3$.

3. (a) $\frac{d}{dx}\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{d}{dx}(1) \Rightarrow -\frac{1}{x^2} - \frac{1}{y^2}y' = 0 \Rightarrow -\frac{1}{y^2}y' = \frac{1}{x^2} \Rightarrow y' = -\frac{y^2}{x^2}$

(b) $\frac{1}{x} + \frac{1}{y} = 1 \Rightarrow \frac{1}{y} = 1 - \frac{1}{x} = \frac{x-1}{x} \Rightarrow y = \frac{x}{x-1}$, so $y' = \frac{(x-1)(1) - (x)(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$.

(c) $y' = -\frac{y^2}{x^2} = -\frac{[x/(x-1)]^2}{x^2} = -\frac{x^2}{x^2(x-1)^2} = -\frac{1}{(x-1)^2}$

5. $\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(1) \Rightarrow 3x^2 + 3y^2 \cdot y' = 0 \Rightarrow 3y^2 y' = -3x^2 \Rightarrow y' = -\frac{x^2}{y^2}$

7. $\frac{d}{dx}(x^2 + xy - y^2) = \frac{d}{dx}(4) \Rightarrow 2x + x \cdot y' + y \cdot 1 - 2y\,y' = 0 \Rightarrow$
$xy' - 2y\,y' = -2x - y \Rightarrow (x - 2y)\,y' = -2x - y \Rightarrow y' = \frac{-2x-y}{x-2y} = \frac{2x+y}{2y-x}$

9. $\frac{d}{dx}[x^4(x+y)] = \frac{d}{dx}[y^2(3x-y)] \Rightarrow x^4(1+y') + (x+y) \cdot 4x^3 = y^2(3-y') + (3x-y) \cdot 2y\,y' \Rightarrow$
$x^4 + x^4 y' + 4x^4 + 4x^3 y = 3y^2 - y^2 y' + 6xy\,y' - 2y^2 y' \Rightarrow x^4 y' + 3y^2 y' - 6xy\,y' = 3y^2 - 5x^4 - 4x^3 y \Rightarrow$
$(x^4 + 3y^2 - 6xy)\,y' = 3y^2 - 5x^4 - 4x^3 y \Rightarrow y' = \frac{3y^2 - 5x^4 - 4x^3 y}{x^4 + 3y^2 - 6xy}$

11. $\frac{d}{dx}(x^2y^2 + x\sin y) = \frac{d}{dx}(4) \Rightarrow x^2 \cdot 2y\,y' + y^2 \cdot 2x + x\cos y \cdot y' + \sin y \cdot 1 = 0 \Rightarrow$
$2x^2 y\,y' + x\cos y \cdot y' = -2xy^2 - \sin y \Rightarrow (2x^2 y + x\cos y)y' = -2xy^2 - \sin y \Rightarrow y' = \frac{-2xy^2 - \sin y}{2x^2 y + x\cos y}$

13. $\frac{d}{dx}(4\cos x\,\sin y) = \frac{d}{dx}(1) \Rightarrow 4[\cos x \cdot \cos y \cdot y' + \sin y \cdot (-\sin x)] = 0 \Rightarrow$
$y'(4\cos x\,\cos y) = 4\sin x\,\sin y \Rightarrow y' = \frac{4\sin x\,\sin y}{4\cos x\,\cos y} = \tan x\,\tan y$

15. $\frac{d}{dx}(e^{x/y}) = \frac{d}{dx}(x - y) \Rightarrow e^{x/y} \cdot \frac{d}{dx}\left(\frac{x}{y}\right) = 1 - y' \Rightarrow$
$e^{x/y} \cdot \frac{y \cdot 1 - x \cdot y'}{y^2} = 1 - y' \Rightarrow e^{x/y} \cdot \frac{1}{y} - \frac{xe^{x/y}}{y^2} \cdot y' = 1 - y' \Rightarrow y' - \frac{xe^{x/y}}{y^2} \cdot y' = 1 - \frac{e^{x/y}}{y} \Rightarrow$
$y'\left(1 - \frac{xe^{x/y}}{y^2}\right) = \frac{y - e^{x/y}}{y} \Rightarrow y' = \frac{\frac{y - e^{x/y}}{y}}{\frac{y^2 - xe^{x/y}}{y^2}} = \frac{y(y - e^{x/y})}{y^2 - xe^{x/y}}$

17. $\frac{d}{dx}\tan^{-1}(x^2y) = \frac{d}{dx}(x + xy^2) \Rightarrow \frac{1}{1+(x^2y)^2}(x^2y' + y\cdot 2x) = 1 + x\cdot 2y\,y' + y^2\cdot 1 \Rightarrow$

$\frac{x^2}{1+x^4y^2}\,y' - 2xy\,y' = 1 + y^2 - \frac{2xy}{1+x^4y^2} \Rightarrow y'\left(\frac{x^2}{1+x^4y^2} - 2xy\right) = 1 + y^2 - \frac{2xy}{1+x^4y^2} \Rightarrow$

$$y' = \frac{1 + y^2 - \dfrac{2xy}{1+x^4y^2}}{\dfrac{x^2}{1+x^4y^2} - 2xy} \quad \text{or} \quad y' = \frac{1 + x^4y^2 + y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}$$

19. $\frac{d}{dx}(e^y\cos x) = \frac{d}{dx}[1 + \sin(xy)] \Rightarrow e^y(-\sin x) + \cos x\cdot e^y\cdot y' = \cos(xy)\cdot(xy' + y\cdot 1) \Rightarrow$

$-e^y\sin x + e^y\cos x\cdot y' = x\cos(xy)\cdot y' + y\cos(xy) \Rightarrow e^y\cos x\cdot y' - x\cos(xy)\cdot y' = e^y\sin x + y\cos(xy) \Rightarrow$

$[e^y\cos x - x\cos(xy)]\,y' = e^y\sin x + y\cos(xy) \Rightarrow y' = \frac{e^y\sin x + y\cos(xy)}{e^y\cos x - x\cos(xy)}$

21. $\frac{d}{dx}\{f(x) + x^2[f(x)]^3\} = \frac{d}{dx}(10) \Rightarrow f'(x) + x^2\cdot 3[f(x)]^2\cdot f'(x) + [f(x)]^3\cdot 2x = 0$. If $x = 1$, we have

$f'(1) + 1^2\cdot 3[f(1)]^2\cdot f'(1) + [f(1)]^3\cdot 2(1) = 0 \Rightarrow f'(1) + 1\cdot 3\cdot 2^2\cdot f'(1) + 2^3\cdot 2 = 0 \Rightarrow$

$f'(1) + 12f'(1) = -16 \Rightarrow 13f'(1) = -16 \Rightarrow f'(1) = -\frac{16}{13}$.

23. $\frac{d}{dy}(x^4y^2 - x^3y + 2xy^3) = \frac{d}{dy}(0) \Rightarrow x^4\cdot 2y + y^2\cdot 4x^3\,x' - (x^3\cdot 1 + y\cdot 3x^2\,x') + 2(x\cdot 3y^2 + y^3\cdot x') = 0 \Rightarrow$

$4x^3y^2\,x' - 3x^2y\,x' + 2y^3\,x' = -2x^4y + x^3 - 6xy^2 \Rightarrow (4x^3y^2 - 3x^2y + 2y^3)\,x' = -2x^4y + x^3 - 6xy^2 \Rightarrow$

$x' = \frac{dx}{dy} = \frac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$

25. $y\sin 2x = x\cos 2y \Rightarrow y\cdot\cos 2x\cdot 2 + \sin 2x\cdot y' = x(-\sin 2y\cdot 2y') + \cos(2y)\cdot 1 \Rightarrow$

$\sin 2x\cdot y' + 2x\sin 2y\cdot y' = -2y\cos 2x + \cos 2y \Rightarrow$

$y'(\sin 2x + 2x\sin 2y) = -2y\cos 2x + \cos 2y \Rightarrow y' = \frac{-2y\cos 2x + \cos 2y}{\sin 2x + 2x\sin 2y}$. When $x = \frac{\pi}{2}$ and $y = \frac{\pi}{4}$, we have

$y' = \frac{(-\pi/2)(-1) + 0}{0 + \pi\cdot 1} = \frac{\pi/2}{\pi} = \frac{1}{2}$, so an equation of the tangent line is $y - \frac{\pi}{4} = \frac{1}{2}(x - \frac{\pi}{2})$, or $y = \frac{1}{2}x$.

27. $x^2 + xy + y^2 = 3 \Rightarrow 2x + x\,y' + y\cdot 1 + 2yy' = 0 \Rightarrow x\,y' + 2y\,y' = -2x - y \Rightarrow y'(x + 2y) = -2x - y \Rightarrow$

$y' = \frac{-2x - y}{x + 2y}$. When $x = 1$ and $y = 1$, we have $y' = \frac{-2 - 1}{1 + 2\cdot 1} = \frac{-3}{3} = -1$, so an equation of the tangent line is

$y - 1 = -1(x - 1)$ or $y = -x + 2$.

29. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \Rightarrow 2x + 2y\,y' = 2(2x^2 + 2y^2 - x)(4x + 4y\,y' - 1)$. When $x = 0$ and $y = \frac{1}{2}$, we have

$0 + y' = 2(\frac{1}{2})(2y' - 1) \Rightarrow y' = 2y' - 1 \Rightarrow y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x - 0)$

or $y = x + \frac{1}{2}$.

31. $2(x^2+y^2)^2 = 25(x^2-y^2) \Rightarrow 4(x^2+y^2)(2x+2y\,y') = 25(2x-2y\,y') \Rightarrow$

$4(x+y\,y')(x^2+y^2) = 25(x-y\,y') \Rightarrow 4y\,y'(x^2+y^2)+25yy' = 25x-4x(x^2+y^2) \Rightarrow$

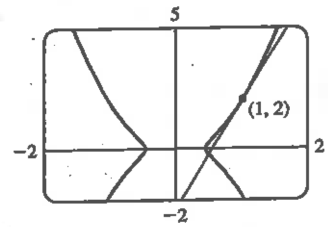
$y' = \dfrac{25x-4x(x^2+y^2)}{25y+4y(x^2+y^2)}$. When $x=3$ and $y=1$, we have $y' = \frac{75-120}{25+40} = -\frac{45}{65} = -\frac{9}{13}$,

so an equation of the tangent line is $y-1 = -\frac{9}{13}(x-3)$ or $y = -\frac{9}{13}x+\frac{40}{13}$.

33. (a) $y^2 = 5x^4-x^2 \Rightarrow 2y\,y' = 5(4x^3)-2x \Rightarrow y' = \dfrac{10x^3-x}{y}$.

So at the point $(1,2)$ we have $y' = \dfrac{10(1)^3-1}{2} = \dfrac{9}{2}$, and an equation of the tangent line is $y-2 = \frac{9}{2}(x-1)$ or $y = \frac{9}{2}x-\frac{5}{2}$.

(b)



35. $9x^2+y^2 = 9 \Rightarrow 18x+2y\,y' = 0 \Rightarrow 2y\,y' = -18x \Rightarrow y' = -9x/y \Rightarrow$

$y'' = -9\left(\dfrac{y\cdot 1 - x\cdot y'}{y^2}\right) = -9\left(\dfrac{y-x(-9x/y)}{y^2}\right) = -9\cdot\dfrac{y^2+9x^2}{y^3} = -9\cdot\dfrac{9}{y^3}$ [since x and y must satisfy the original equation, $9x^2+y^2=9$]. Thus, $y'' = -81/y^3$.

37. $x^3+y^3 = 1 \Rightarrow 3x^2+3y^2\,y' = 0 \Rightarrow y' = -\dfrac{x^2}{y^2} \Rightarrow$

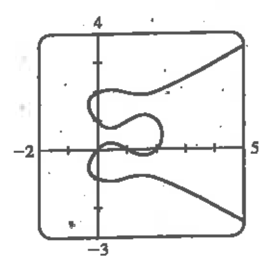
$y'' = -\dfrac{y^2(2x)-x^2\cdot 2y\,y'}{(y^2)^2} = -\dfrac{2xy^2-2x^2y(-x^2/y^2)}{y^4} = -\dfrac{2xy^4+2x^4y}{y^6} = -\dfrac{2xy(y^3+x^3)}{y^6} = -\dfrac{2x}{y^5}$,

since x and y must satisfy the original equation, $x^3+y^3=1$.

39. If $x=0$ in $xy+e^y=e$, then we get $0+e^y=e$, so $y=1$ and the point where $x=0$ is $(0,1)$. Differentiating implicitly with respect to x gives us $xy'+y\cdot 1+e^y y' = 0$. Substituting 0 for x and 1 for y gives us $0+1+ey'=0 \Rightarrow ey'=-1 \Rightarrow y'=-1/e$. Differentiating $xy'+y+e^y y'=0$ implicitly with respect to x gives us $xy''+y'\cdot 1+y'+e^y y''+y'\cdot e^y y' = 0$. Now substitute 0 for x, 1 for y, and $-1/e$ for y'.

$0+\left(-\dfrac{1}{e}\right)+\left(-\dfrac{1}{e}\right)+ey''+\left(-\dfrac{1}{e}\right)(e)\left(-\dfrac{1}{e}\right) = 0 \Rightarrow -\dfrac{2}{e}+ey''+\dfrac{1}{e} = 0 \Rightarrow ey'' = \dfrac{1}{e} \Rightarrow y'' = \dfrac{1}{e^2}$.

41. (a) There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

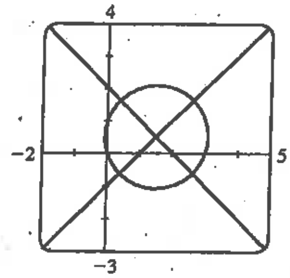


(b) $y' = \dfrac{3x^2-6x+2}{2(2y^3-3y^2-y+1)} \Rightarrow y'=-1$ at $(0,1)$ and $y'=\frac{1}{3}$ at $(0,2)$.

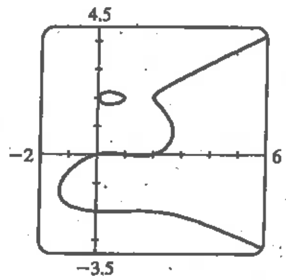
Equations of the tangent lines are $y=-x+1$ and $y=\frac{1}{3}x+2$.

(c) $y'=0 \Rightarrow 3x^2-6x+2=0 \Rightarrow x = 1\pm\frac{1}{3}\sqrt{3}$

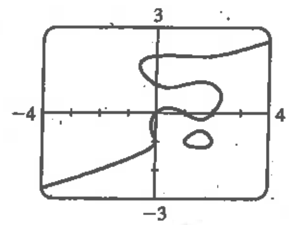
(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph. By modifying the equation in other ways, we can generate the other graphs.



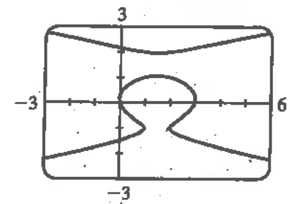
$y(y^2 - 1)(y - 2)$
$= x(x - 1)(x - 2)(x - 3)$



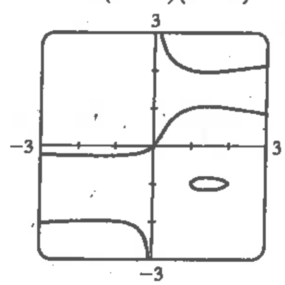
$y(y^2 - 4)(y - 2)$
$= x(x - 1)(x - 2)$



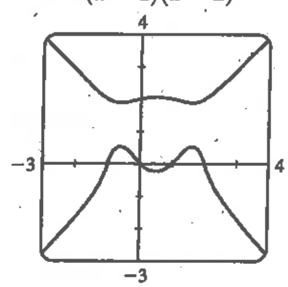
$y(y + 1)(y^2 - 1)(y - 2)$
$= x(x - 1)(x - 2)$



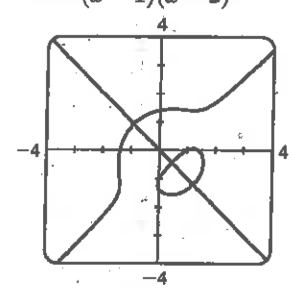
$(y + 1)(y^2 - 1)(y - 2)$
$= (x - 1)(x - 2)$



$x(y + 1)(y^2 - 1)(y - 2)$
$= y(x - 1)(x - 2)$



$y(y^2 + 1)(y - 2)$
$= x(x^2 - 1)(x - 2)$



$y(y + 1)(y^2 - 2)$
$= x(x - 1)(x^2 - 2)$

43. From Exercise 31, a tangent to the lemniscate will be horizontal if $y' = 0 \Rightarrow 25x - 4x(x^2 + y^2) = 0 \Rightarrow$ $x[25 - 4(x^2 + y^2)] = 0 \Rightarrow x^2 + y^2 = \frac{25}{4}$ (1). (Note that when x is 0, y is also 0, and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for $x^2 + y^2$ in the equation of the lemniscate, $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, we get $x^2 - y^2 = \frac{25}{8}$ (2). Solving (1) and (2), we have $x^2 = \frac{75}{16}$ and $y^2 = \frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4}, \pm\frac{5}{4}\right)$.

45. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{b^2x_0}{a^2y_0}(x - x_0)$. Multiplying both sides by $\frac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = \dfrac{x_0x}{a^2} - \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the hyperbola, we have $\dfrac{x_0x}{a^2} - \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$.