

57. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \Rightarrow$

$$\begin{aligned} T'(1999) &= P(1999)A'(1999) + A(1999)P'(1999) = (961,400)(\$1400/\text{yr}) + (\$30,593)(9200/\text{yr}) \\ &= \$1,345,960,000/\text{yr} + \$281,455,600/\text{yr} = \$1,627,415,600/\text{yr} \end{aligned}$$

So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

59. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\frac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x)$.

(c) $\frac{d}{dx}(e^{3x}) = \frac{d}{dx}(e^x)^3 = 3(e^x)^2 e^x = 3e^{2x} e^x = 3e^{3x}$

61. For $f(x) = x^2 e^x$, $f'(x) = x^2 e^x + e^x(2x) = e^x(x^2 + 2x)$. Similarly, we have

$$f''(x) = e^x(x^2 + 4x + 2)$$

$$f'''(x) = e^x(x^2 + 6x + 6)$$

$$f^{(4)}(x) = e^x(x^2 + 8x + 12)$$

$$f^{(5)}(x) = e^x(x^2 + 10x + 20)$$

It appears that the coefficient of x in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0 = 1 \cdot 0$, $2 = 2 \cdot 1$, $6 = 3 \cdot 2$, $12 = 4 \cdot 3$, $20 = 5 \cdot 4$. So a reasonable guess is that

$$f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)].$$

Proof: Let S_n be the statement that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

1. S_1 is true because $f'(x) = e^x(x^2 + 2x)$.

2. Assume that S_k is true; that is, $f^{(k)}(x) = e^x[x^2 + 2kx + k(k-1)]$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx} [f^{(k)}(x)] = e^x(2x + 2k) + [x^2 + 2kx + k(k-1)]e^x \\ &= e^x[x^2 + (2k+2)x + (k^2+k)] = e^x[x^2 + 2(k+1)x + (k+1)k] \end{aligned}$$

This shows that S_{k+1} is true.

3. Therefore, by mathematical induction, S_n is true for all n ; that is, $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$ for every positive integer n .

3.3 Derivatives of Trigonometric Functions

1. $f(x) = 3x^2 - 2 \cos x \Rightarrow f'(x) = 6x - 2(-\sin x) = 6x + 2 \sin x$

3. $f(x) = \sin x + \frac{1}{2} \cot x \Rightarrow f'(x) = \cos x - \frac{1}{2} \csc^2 x$

$$5. g(t) = t^3 \cos t \Rightarrow g'(t) = t^3(-\sin t) + (\cos t) \cdot 3t^2 = 3t^2 \cos t - t^3 \sin t \text{ or } t^2(3 \cos t - t \sin t)$$

$$7. h(\theta) = \csc \theta + e^\theta \cot \theta \Rightarrow h'(\theta) = -\csc \theta \cot \theta + e^\theta(-\csc^2 \theta) + (\cot \theta)e^\theta = -\csc \theta \cot \theta + e^\theta(\cot \theta - \csc^2 \theta)$$

$$9. y = \frac{x}{2 - \tan x} \Rightarrow y' = \frac{(2 - \tan x)(1) - x(-\sec^2 x)}{(2 - \tan x)^2} = \frac{2 - \tan x + x \sec^2 x}{(2 - \tan x)^2}$$

$$11. f(\theta) = \frac{\sec \theta}{1 + \sec \theta} \Rightarrow$$

$$f'(\theta) = \frac{(1 + \sec \theta)(\sec \theta \tan \theta) - (\sec \theta)(\sec \theta \tan \theta)}{(1 + \sec \theta)^2} = \frac{(\sec \theta \tan \theta)[(1 + \sec \theta) - \sec \theta]}{(1 + \sec \theta)^2} = \frac{\sec \theta \tan \theta}{(1 + \sec \theta)^2}$$

$$13. y = \frac{t \sin t}{1 + t} \Rightarrow$$

$$y' = \frac{(1 + t)(t \cos t + \sin t) - t \sin t(1)}{(1 + t)^2} = \frac{t \cos t + \sin t + t^2 \cos t + t \sin t - t \sin t}{(1 + t)^2} = \frac{(t^2 + t) \cos t + \sin t}{(1 + t)^2}$$

$$15. \text{ Using Exercise 3.2.59(a), } f(x) = x e^x \csc x \Rightarrow$$

$$f'(x) = (x)' e^x \csc x + x(e^x)' \csc x + x e^x (\csc x)' = 1 e^x \csc x + x e^x \csc x + x e^x (-\cot x \csc x) \\ = e^x \csc x (1 + x - x \cot x)$$

$$17. \frac{d}{dx}(\csc x) = \frac{d}{dx} \left(\frac{1}{\sin x} \right) = \frac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

$$19. \frac{d}{dx}(\cot x) = \frac{d}{dx} \left(\frac{\cos x}{\sin x} \right) = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

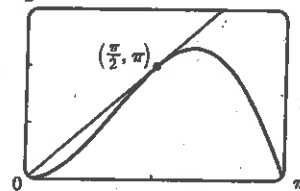
$$21. y = \sec x \Rightarrow y' = \sec x \tan x, \text{ so } y'(\frac{\pi}{3}) = \sec \frac{\pi}{3} \tan \frac{\pi}{3} = 2\sqrt{3}. \text{ An equation of the tangent line to the curve } y = \sec x \\ \text{ at the point } (\frac{\pi}{3}, 2) \text{ is } y - 2 = 2\sqrt{3}(x - \frac{\pi}{3}) \text{ or } y = 2\sqrt{3}x + 2 - \frac{2}{3}\sqrt{3}\pi.$$

$$23. y = \cos x - \sin x \Rightarrow y' = -\sin x - \cos x, \text{ so } y'(\pi) = -\sin \pi - \cos \pi = 0 - (-1) = 1. \text{ An equation of the tangent } \\ \text{ line to the curve } y = \cos x - \sin x \text{ at the point } (\pi, -1) \text{ is } y - (-1) = 1(x - \pi) \text{ or } y = x - \pi - 1.$$

$$25. (a) y = 2x \sin x \Rightarrow y' = 2(x \cos x + \sin x \cdot 1). \text{ At } (\frac{\pi}{2}, \pi),$$

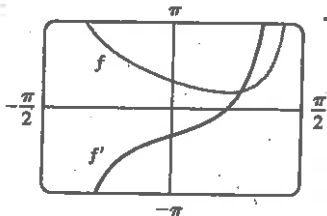
$$y' = 2(\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}) = 2(0 + 1) = 2, \text{ and an equation of the } \\ \text{ tangent line is } y - \pi = 2(x - \frac{\pi}{2}), \text{ or } y = 2x.$$

$$(b) \frac{3\pi}{2}$$



$$27. (a) f(x) = \sec x - x \Rightarrow f'(x) = \sec x \tan x - 1$$

(b)



Note that $f' = 0$ where f has a minimum. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

$$29. H(\theta) = \theta \sin \theta \Rightarrow H'(\theta) = \theta (\cos \theta) + (\sin \theta) \cdot 1 = \theta \cos \theta + \sin \theta \Rightarrow$$

$$H''(\theta) = \theta (-\sin \theta) + (\cos \theta) \cdot 1 + \cos \theta = -\theta \sin \theta + 2 \cos \theta$$

$$31. (a) f(x) = \frac{\tan x - 1}{\sec x} \Rightarrow$$

$$f'(x) = \frac{\sec x (\sec^2 x) - (\tan x - 1)(\sec x \tan x)}{(\sec x)^2} = \frac{\sec x (\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

$$(b) f(x) = \frac{\tan x - 1}{\sec x} = \frac{\frac{\sin x}{\cos x} - 1}{\frac{1}{\cos x}} = \frac{\sin x - \cos x}{\frac{1}{\cos x}} = \sin x - \cos x \Rightarrow f'(x) = \cos x - (-\sin x) = \cos x + \sin x$$

$$(c) \text{ From part (a), } f'(x) = \frac{1 + \tan x}{\sec x} = \frac{1}{\sec x} + \frac{\tan x}{\sec x} = \cos x + \sin x, \text{ which is the expression for } f'(x) \text{ in part (b).}$$

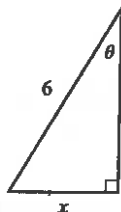
$$33. f(x) = x + 2 \sin x \text{ has a horizontal tangent when } f'(x) = 0 \Leftrightarrow 1 + 2 \cos x = 0 \Leftrightarrow \cos x = -\frac{1}{2} \Leftrightarrow$$

$x = \frac{2\pi}{3} + 2\pi n \text{ or } \frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm \frac{\pi}{3}$ units from π . This allows us to write the solutions in the more compact equivalent form $(2n + 1)\pi \pm \frac{\pi}{3}$, n an integer.

$$35. (a) x(t) = 8 \sin t \Rightarrow v(t) = x'(t) = 8 \cos t \Rightarrow a(t) = x''(t) = -8 \sin t$$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x(\frac{2\pi}{3}) = 8 \sin \frac{2\pi}{3} = 8(\frac{\sqrt{3}}{2}) = 4\sqrt{3}$, velocity $v(\frac{2\pi}{3}) = 8 \cos \frac{2\pi}{3} = 8(-\frac{1}{2}) = -4$, and acceleration $a(\frac{2\pi}{3}) = -8 \sin \frac{2\pi}{3} = -8(\frac{\sqrt{3}}{2}) = -4\sqrt{3}$. Since $v(\frac{2\pi}{3}) < 0$, the particle is moving to the left.

37.



From the diagram we can see that $\sin \theta = x/6 \Leftrightarrow x = 6 \sin \theta$. We want to find the rate of change of x with respect to θ ; that is, $dx/d\theta$. Taking the derivative of the above expression, $dx/d\theta = 6(\cos \theta)$. So when $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 6 \cos \frac{\pi}{3} = 6(\frac{1}{2}) = 3$ m/rad.

$$39. \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \lim_{x \rightarrow 0} \frac{3 \sin 3x}{3x} \quad [\text{multiply numerator and denominator by 3}]$$

$$= 3 \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x} \quad [\text{as } x \rightarrow 0, 3x \rightarrow 0]$$

$$= 3 \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \quad [\text{let } \theta = 3x]$$

$$= 3(1) \quad [\text{Equation 2}]$$

$$= 3$$

$$41. \lim_{t \rightarrow 0} \frac{\tan 6t}{\sin 2t} = \lim_{t \rightarrow 0} \left(\frac{\sin 6t}{t} \cdot \frac{1}{\cos 6t} \cdot \frac{t}{\sin 2t} \right) = \lim_{t \rightarrow 0} \frac{6 \sin 6t}{6t} \cdot \lim_{t \rightarrow 0} \frac{1}{\cos 6t} \cdot \lim_{t \rightarrow 0} \frac{2t}{2 \sin 2t}$$

$$= 6 \lim_{t \rightarrow 0} \frac{\sin 6t}{6t} \cdot \lim_{t \rightarrow 0} \frac{1}{\cos 6t} \cdot \frac{1}{2} \lim_{t \rightarrow 0} \frac{2t}{\sin 2t} = 6(1) \cdot \frac{1}{1} \cdot \frac{1}{2}(1) = 3$$

$$43. \lim_{x \rightarrow 0} \frac{\sin 3x}{5x^3 - 4x} = \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{3x} \cdot \frac{3}{5x^2 - 4} \right) = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \lim_{x \rightarrow 0} \frac{3}{5x^2 - 4} = 1 \cdot \left(\frac{3}{-4} \right) = -\frac{3}{4}$$

45. Divide numerator and denominator by θ . ($\sin \theta$ also works.)

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta + \tan \theta} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\theta}}{1 + \frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta}} = \frac{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}}{1 + \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta}} = \frac{1}{1 + 1 \cdot 1} = \frac{1}{2}$$

$$47. \lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\sin x - \cos x} = \lim_{x \rightarrow \pi/4} \frac{\left(1 - \frac{\sin x}{\cos x}\right) \cdot \cos x}{(\sin x - \cos x) \cdot \cos x} = \lim_{x \rightarrow \pi/4} \frac{\cos x - \sin x}{(\sin x - \cos x) \cos x} = \lim_{x \rightarrow \pi/4} \frac{-1}{\cos x} = \frac{-1}{1/\sqrt{2}} = -\sqrt{2}$$

$$49. \frac{d}{dx}(\sin x) = \cos x \Rightarrow \frac{d^2}{dx^2}(\sin x) = -\sin x \Rightarrow \frac{d^3}{dx^3}(\sin x) = -\cos x \Rightarrow \frac{d^4}{dx^4}(\sin x) = \sin x.$$

The derivatives of $\sin x$ occur in a cycle of four. Since $99 = 4(24) + 3$, we have $\frac{d^{99}}{dx^{99}}(\sin x) = \frac{d^3}{dx^3}(\sin x) = -\cos x$.

51. $y = A \sin x + B \cos x \Rightarrow y' = A \cos x - B \sin x \Rightarrow y'' = -A \sin x - B \cos x$. Substituting these expressions for y , y' , and y'' into the given differential equation $y'' + y' - 2y = \sin x$ gives us

$$(-A \sin x - B \cos x) + (A \cos x - B \sin x) - 2(A \sin x + B \cos x) = \sin x \Leftrightarrow$$

$$-3A \sin x - B \sin x + A \cos x - 3B \cos x = \sin x \Leftrightarrow (-3A - B) \sin x + (A - 3B) \cos x = 1 \sin x, \text{ so we must have}$$

$-3A - B = 1$ and $A - 3B = 0$ (since 0 is the coefficient of $\cos x$ on the right side). Solving for A and B , we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

$$53. (a) \frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} \Rightarrow \sec^2 x = \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}. \text{ So } \sec^2 x = \frac{1}{\cos^2 x}.$$

$$(b) \frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} \Rightarrow \sec x \tan x = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}. \text{ So } \sec x \tan x = \frac{\sin x}{\cos^2 x}.$$

$$(c) \frac{d}{dx}(\sin x + \cos x) = \frac{d}{dx} \frac{1 + \cot x}{\csc x} \Rightarrow$$

$$\begin{aligned} \cos x - \sin x &= \frac{\csc x (-\csc^2 x) - (1 + \cot x)(-\csc x \cot x)}{\csc^2 x} = \frac{\csc x [-\csc^2 x + (1 + \cot x) \cot x]}{\csc^2 x} \\ &= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1 + \cot x}{\csc x} \end{aligned}$$

$$\text{So } \cos x - \sin x = \frac{\cot x - 1}{\csc x}.$$

55. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle. By drawing the bisector of the angle θ , we can

$$\text{see that } \sin \frac{\theta}{2} = \frac{d/2}{r} \Rightarrow d = 2r \sin \frac{\theta}{2}. \text{ So } \lim_{\theta \rightarrow 0^+} \frac{s}{d} = \lim_{\theta \rightarrow 0^+} \frac{r\theta}{2r \sin(\theta/2)} = \lim_{\theta \rightarrow 0^+} \frac{2 \cdot (\theta/2)}{2 \sin(\theta/2)} = \lim_{\theta \rightarrow 0} \frac{\theta/2}{\sin(\theta/2)} = 1.$$

[This is just the reciprocal of the limit $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ combined with the fact that as $\theta \rightarrow 0$, $\frac{\theta}{2} \rightarrow 0$ also.]