## 3.2 The Product and Quotient Rules

**1.** Product Rule: $f(x) = (1+2x^2)(x-x^2) \Rightarrow$

$$f'(x) = (1+2x^2)(1-2x) + (x-x^2)(4x) = 1-2x+2x^2-4x^3+4x^2-4x^3 = 1-2x+6x^2-8x^3.$$

Multiplying first: $f(x) = (1+2x^2)(x-x^2) = x - x^2 + 2x^3 - 2x^4 \Rightarrow f'(x) = 1 - 2x + 6x^2 - 8x^3$ (equivalent).

**3.** By the Product Rule, $f(x) = (x^3+2x)e^x \Rightarrow$

$$\begin{aligned} f'(x) &= (x^3+2x)(e^x)' + e^x(x^3+2x)' = (x^3+2x)e^x + e^x(3x^2+2) \\ &= e^x[(x^3+2x)+(3x^2+2)] = e^x(x^3+3x^2+2x+2) \end{aligned}$$

**5.** By the Quotient Rule, $y = \dfrac{e^x}{x^2} \Rightarrow y' = \dfrac{x^2 \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(x^2)}{(x^2)^2} = \dfrac{x^2(e^x) - e^x(2x)}{x^4} = \dfrac{xe^x(x-2)}{x^4} = \dfrac{e^x(x-2)}{x^3}.$

The notations $\overset{\text{PR}}{\Rightarrow}$ and $\overset{\text{QR}}{\Rightarrow}$ indicate the use of the Product and Quotient Rules, respectively.

**7.** $g(x) = \dfrac{3x-1}{2x+1} \overset{\text{QR}}{\Rightarrow} g'(x) = \dfrac{(2x+1)(3)-(3x-1)(2)}{(2x+1)^2} = \dfrac{6x+3-6x+2}{(2x+1)^2} = \dfrac{5}{(2x+1)^2}$

**9.** $H(u) = (u-\sqrt{u})(u+\sqrt{u}) \overset{\text{PR}}{\Rightarrow}$

$$H'(u) = (u-\sqrt{u})\left(1+\frac{1}{2\sqrt{u}}\right) + (u+\sqrt{u})\left(1-\frac{1}{2\sqrt{u}}\right) = u + \tfrac{1}{2}\sqrt{u} - \sqrt{u} - \tfrac{1}{2} + u - \tfrac{1}{2}\sqrt{u} + \sqrt{u} - \tfrac{1}{2} = 2u - 1.$$

An easier method is to simplify first and then differentiate as follows:

$H(u) = (u-\sqrt{u})(u+\sqrt{u}) = u^2 - (\sqrt{u})^2 = u^2 - u \Rightarrow H'(u) = 2u - 1$

**11.** $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y+5y^3) = (y^{-2}-3y^{-4})(y+5y^3) \overset{\text{PR}}{\Rightarrow}$

$$\begin{aligned} F'(y) &= (y^{-2}-3y^{-4})(1+15y^2) + (y+5y^3)(-2y^{-3}+12y^{-5}) \\ &= (y^{-2}+15-3y^{-4}-45y^{-2}) + (-2y^{-2}+12y^{-4}-10+60y^{-2}) \\ &= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4 \end{aligned}$$

**13.** $y = \dfrac{x^3}{1-x^2} \overset{\text{QR}}{\Rightarrow} y' = \dfrac{(1-x^2)(3x^2) - x^3(-2x)}{(1-x^2)^2} = \dfrac{x^2(3-3x^2+2x^2)}{(1-x^2)^2} = \dfrac{x^2(3-x^2)}{(1-x^2)^2}$

**15.** $y = \dfrac{t^2+2}{t^4-3t^2+1} \overset{\text{QR}}{\Rightarrow}$

$$\begin{aligned} y' &= \frac{(t^4-3t^2+1)(2t) - (t^2+2)(4t^3-6t)}{(t^4-3t^2+1)^2} = \frac{2t[(t^4-3t^2+1)-(t^2+2)(2t^2-3)]}{(t^4-3t^2+1)^2} \\ &= \frac{2t(t^4-3t^2+1-2t^4-4t^2+3t^2+6)}{(t^4-3t^2+1)^2} = \frac{2t(-t^4-4t^2+7)}{(t^4-3t^2+1)^2} \end{aligned}$$

**17.** $y = e^p(p+p\sqrt{p}) = e^p(p+p^{3/2}) \overset{\text{PR}}{\Rightarrow} y' = e^p\left(1+\tfrac{3}{2}p^{1/2}\right) + (p+p^{3/2})e^p = e^p\left(1+\tfrac{3}{2}\sqrt{p}+p+p\sqrt{p}\right)$

**19.** $y = \dfrac{v^3 - 2v\sqrt{v}}{v} = v^2 - 2\sqrt{v} = v^2 - 2v^{1/2} \quad\Rightarrow\quad y' = 2v - 2\left(\tfrac{1}{2}\right)v^{-1/2} = 2v - v^{-1/2}.$

We can change the form of the answer as follows: $2v - v^{-1/2} = 2v - \dfrac{1}{\sqrt{v}} = \dfrac{2v\sqrt{v} - 1}{\sqrt{v}} = \dfrac{2v^{3/2} - 1}{\sqrt{v}}$

**21.** $f(t) = \dfrac{2t}{2+\sqrt{t}} \quad\overset{\text{QR}}{\Rightarrow}\quad f'(t) = \dfrac{(2+t^{1/2})(2) - 2t\left(\frac{1}{2}t^{-1/2}\right)}{(2+\sqrt{t})^2} = \dfrac{4 + 2t^{1/2} - t^{1/2}}{(2+\sqrt{t})^2} = \dfrac{4 + t^{1/2}}{(2+\sqrt{t})^2}$ or $\dfrac{4+\sqrt{t}}{(2+\sqrt{t})^2}$

**23.** $f(x) = \dfrac{A}{B + Ce^x} \quad\overset{\text{QR}}{\Rightarrow}\quad f'(x) = \dfrac{(B + Ce^x)\cdot 0 - A(Ce^x)}{(B+Ce^x)^2} = -\dfrac{ACe^x}{(B+Ce^x)^2}$

**25.** $f(x) = \dfrac{x}{x + c/x} \quad\Rightarrow\quad f'(x) = \dfrac{(x + c/x)(1) - x(1 - c/x^2)}{\left(x + \frac{c}{x}\right)^2} = \dfrac{x + c/x - x + c/x}{\left(\frac{x^2+c}{x}\right)^2} = \dfrac{2c/x}{\frac{(x^2+c)^2}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{2cx}{(x^2+c)^2}$

**27.** $f(x) = x^4e^x \quad\Rightarrow\quad f'(x) = x^4e^x + e^x\cdot 4x^3 = (x^4 + 4x^3)e^x \quad [\text{or } x^3e^x(x+4)] \quad\Rightarrow$

$$f''(x) = (x^4 + 4x^3)e^x + e^x(4x^3 + 12x^2) = (x^4 + 4x^3 + 4x^3 + 12x^2)e^x$$
$$= (x^4 + 8x^3 + 12x^2)e^x \quad [\text{or } x^2e^x(x+2)(x+6)]$$

**29.** $f(x) = \dfrac{x^2}{1+2x} \quad\Rightarrow\quad f'(x) = \dfrac{(1+2x)(2x) - x^2(2)}{(1+2x)^2} = \dfrac{2x + 4x^2 - 2x^2}{(1+2x)^2} = \dfrac{2x^2 + 2x}{(1+2x)^2} \quad\Rightarrow$

$$f''(x) = \frac{(1+2x)^2(4x+2) - (2x^2+2x)(1+4x+4x^2)'}{[(1+2x)^2]^2} = \frac{2(1+2x)^2(2x+1) - 2x(x+1)(4+8x)}{(1+2x)^4}$$
$$= \frac{2(1+2x)[(1+2x)^2 - 4x(x+1)]}{(1+2x)^4} = \frac{2(1+4x+4x^2-4x^2-4x)}{(1+2x)^3} = \frac{2}{(1+2x)^3}$$

**31.** $y = \dfrac{x^2 - 1}{x^2 + x + 1} \quad\Rightarrow$

$$y' = \frac{(x^2+x+1)(2x) - (x^2-1)(2x+1)}{(x^2+x+1)^2} = \frac{2x^3 + 2x^2 + 2x - 2x^3 - x^2 + 2x + 1}{(x^2+x+1)^2} = \frac{x^2+4x+1}{(x^2+x+1)^2}.$$

At $(1, 0)$, $y' = \dfrac{6}{3^2} = \dfrac{2}{3}$, and an equation of the tangent line is $y - 0 = \frac{2}{3}(x-1)$, or $y = \frac{2}{3}x - \frac{2}{3}$.

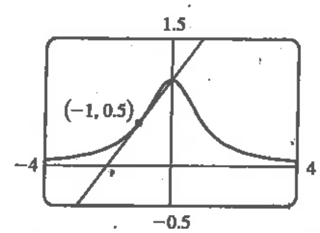
**33.** $y = 2xe^x \quad\Rightarrow\quad y' = 2(x\cdot e^x + e^x\cdot 1) = 2e^x(x+1).$

At $(0, 0)$, $y' = 2e^0(0+1) = 2\cdot 1\cdot 1 = 2$, and an equation of the tangent line is $y - 0 = 2(x - 0)$, or $y = 2x$. The slope of the normal line is $-\frac{1}{2}$, so an equation of the normal line is $y - 0 = -\frac{1}{2}(x - 0)$, or $y = -\frac{1}{2}x$.

**35.** (a) $y = f(x) = \dfrac{1}{1+x^2} \quad\Rightarrow$

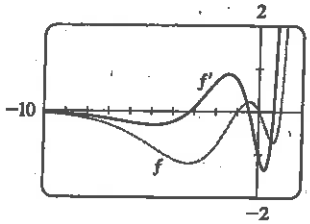
$f'(x) = \dfrac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x + 1$.

(b)



**37.** (a) $f(x) = (x^3 - x)e^x \Rightarrow f'(x) = (x^3 - x)e^x + e^x(3x^2 - 1) = e^x(x^3 + 3x^2 - x - 1)$

(b)




$f' = 0$ when $f$ has a horizontal tangent line, $f'$ is negative when $f$ is decreasing, and $f'$ is positive when $f$ is increasing.

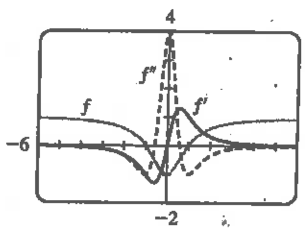
**39.** (a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1} \Rightarrow$

$$f'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{(2x)[(x^2+1) - (x^2-1)]}{(x^2+1)^2} = \frac{(2x)(2)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2} \Rightarrow$$

$$f''(x) = \frac{(x^2+1)^2(4) - 4x(x^4+2x^2+1)'}{[(x^2+1)^2]^2} = \frac{4(x^2+1)^2 - 4x(4x^3+4x)}{(x^2+1)^4}$$

$$= \frac{4(x^2+1)^2 - 16x^2(x^2+1)}{(x^2+1)^4} = \frac{4(x^2+1)[(x^2+1) - 4x^2]}{(x^2+1)^4} = \frac{4(1-3x^2)}{(x^2+1)^3}$$

(b)




$f' = 0$ when $f$ has a horizontal tangent and $f'' = 0$ when $f'$ has a horizontal tangent. $f'$ is negative when $f$ is decreasing and positive when $f$ is increasing. $f''$ is negative when $f'$ is decreasing and positive when $f'$ is increasing. $f''$ is negative when $f$ is concave down and positive when $f$ is concave up.

**41.** $f(x) = \dfrac{x^2}{1+x} \Rightarrow f'(x) = \dfrac{(1+x)(2x) - x^2(1)}{(1+x)^2} = \dfrac{2x + 2x^2 - x^2}{(1+x)^2} = \dfrac{x^2+2x}{x^2+2x+1} \Rightarrow$

$$f''(x) = \frac{(x^2+2x+1)(2x+2) - (x^2+2x)(2x+2)}{(x^2+2x+1)^2} = \frac{(2x+2)(x^2+2x+1-x^2-2x)}{[(x+1)^2]^2}$$

$$= \frac{2(x+1)(1)}{(x+1)^4} = \frac{2}{(x+1)^3},$$

so $f''(1) = \dfrac{2}{(1+1)^3} = \dfrac{2}{8} = \dfrac{1}{4}$.

**43.** We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

**45.** $f(x) = e^x g(x) \Rightarrow f'(x) = e^x g'(x) + g(x)e^x = e^x[g'(x) + g(x)]$. $f'(0) = e^0[g'(0) + g(0)] = 1(5+2) = 7$

**47.** $g(x) = xf(x) \Rightarrow g'(x) = xf'(x) + f(x) \cdot 1$. Now $g(3) = 3f(3) = 3 \cdot 4 = 12$ and $g'(3) = 3f'(3) + f(3) = 3(-2) + 4 = -2$. Thus, an equation of the tangent line to the graph of $g$ at the point where $x = 3$ is $y - 12 = -2(x - 3)$, or $y = -2x + 18$.

**49.** (a) From the graphs of $f$ and $g$, we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of $f$; $g(1) = 1$ since the point $(1, 1)$ is on the graph of $g$; $f'(1) = 2$ since the slope of the line segment between $(0, 0)$ and $(2, 4)$ is $\frac{4-0}{2-0} = 2$; $g'(1) = -1$ since the slope of the line segment between $(-2, 4)$ and $(2, 0)$ is $\frac{0-4}{2-(-2)} = -1$.

Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)f'(1) = 2 \cdot (-1) + 1 \cdot 2 = 0$.

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{2\left(-\frac{1}{3}\right) - 3 \cdot \frac{2}{3}}{2^2} = \dfrac{-\frac{8}{3}}{4} = -\dfrac{2}{3}$

**51.** (a) $y = xg(x) \Rightarrow y' = xg'(x) + g(x) \cdot 1 = xg'(x) + g(x)$.

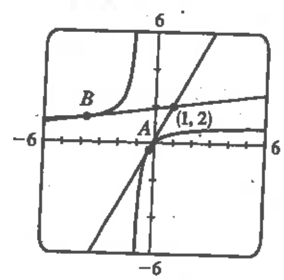
(b) $y = \dfrac{x}{g(x)} \Rightarrow y' = \dfrac{g(x) \cdot 1 - xg'(x)}{[g(x)]^2} = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \dfrac{g(x)}{x} \Rightarrow y' = \dfrac{xg'(x) - g(x) \cdot 1}{(x)^2} = \dfrac{xg'(x) - g(x)}{x^2}$

**53.** If $y = f(x) = \dfrac{x}{x+1}$, then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is $y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x - a)$. This line passes through $(1, 2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1 - a) \Leftrightarrow$ $2(a+1)^2 - a(a+1) = 1 - a \Leftrightarrow 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \Leftrightarrow a^2 + 4a + 1 = 0$.

The quadratic formula gives the roots of this equation as $a = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$, so there are two such tangent lines. Since

$$f(-2 \pm \sqrt{3}) = \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}}$$
$$= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2},$$




the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1-\sqrt{3}}{2}\right) \approx (-0.27, -0.37)$ and $B\left(-2 - \sqrt{3}, \frac{1+\sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.

**55.** $R = \dfrac{f}{g} \Rightarrow R' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = x - 3x^3 + 5x^5$, $f'(x) = 1 - 9x^2 + 25x^4$, and for $g(x) = 1 + 3x^3 + 6x^6 + 9x^9$, $g'(x) = 9x^2 + 36x^5 + 81x^8$.

Thus, $R'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 1 - 0 \cdot 0}{1^2} = \dfrac{1}{1} = 1$.

**57.** If $P(t)$ denotes the population at time $t$ and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \Rightarrow$

$$\begin{aligned} T'(1999) &= P(1999)A'(1999) + A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr}) + (\$30{,}593)(9200/\text{yr}) \\ &= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr} \end{aligned}$$

So the total personal income was rising by about $\$1.627$ billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

**59.** (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\dfrac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x)$.

(c) $\dfrac{d}{dx}\left(e^{3x}\right) = \dfrac{d}{dx}\left(e^x\right)^3 = 3(e^x)^2 e^x = 3e^{2x}e^x = 3e^{3x}$

**61.** For $f(x) = x^2e^x$, $f'(x) = x^2e^x + e^x(2x) = e^x(x^2 + 2x)$. Similarly, we have

$$\begin{aligned} f''(x) &= e^x(x^2 + 4x + 2) \\ f'''(x) &= e^x(x^2 + 6x + 6) \\ f^{(4)}(x) &= e^x(x^2 + 8x + 12) \\ f^{(5)}(x) &= e^x(x^2 + 10x + 20) \end{aligned}$$

It appears that the coefficient of $x$ in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0 = 1 \cdot 0$, $2 = 2 \cdot 1$, $6 = 3 \cdot 2$, $12 = 4 \cdot 3$, $20 = 5 \cdot 4$. So a reasonable guess is that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

*Proof:* Let $S_n$ be the statement that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

1. $S_1$ is true because $f'(x) = e^x(x^2 + 2x)$.
2. Assume that $S_k$ is true; that is, $f^{(k)}(x) = e^x[x^2 + 2kx + k(k-1)]$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}\left[f^{(k)}(x)\right] = e^x(2x + 2k) + [x^2 + 2kx + k(k-1)]e^x \\ &= e^x[x^2 + (2k+2)x + (k^2 + k)] = e^x[x^2 + 2(k+1)x + (k+1)k] \end{aligned}$$

This shows that $S_{k+1}$ is true.

3. Therefore, by mathematical induction, $S_n$ is true for all $n$; that is, $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$ for every positive integer $n$.

## 3.3 Derivatives of Trigonometric Functions

**1.** $f(x) = 3x^2 - 2\cos x \Rightarrow f'(x) = 6x - 2(-\sin x) = 6x + 2\sin x$

**3.** $f(x) = \sin x + \frac{1}{2}\cot x \Rightarrow f'(x) = \cos x - \frac{1}{2}\csc^2 x$