3 □ DIFFERENTIATION RULES

3.1 Derivatives of Polynomials and Exponential Functions

1. (a) e is the number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

(b)

x	$\frac{2.7^x - 1}{x}$
−0.001	0.9928
−0.0001	0.9932
0.001	0.9937
0.0001	0.9933

x	$\frac{2.8^x - 1}{x}$
−0.001	1.0291
−0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places),

$\lim_{h\to 0} \frac{2.7^h - 1}{h} = 0.99$ and $\lim_{h\to 0} \frac{2.8^h - 1}{h} = 1.03$.

Since $0.99 < 1 < 1.03$, $2.7 < e < 2.8$.

3. $f(x) = 186.5$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

5. $f(x) = 5x - 1 \;\Rightarrow\; f'(x) = 5 - 0 = 5$

7. $f(x) = x^3 - 4x + 6 \;\Rightarrow\; f'(x) = 3x^2 - 4(1) + 0 = 3x^2 - 4$

9. $g(x) = x^2(1 - 2x) = x^2 - 2x^3 \;\Rightarrow\; g'(x) = 2x - 2(3x^2) = 2x - 6x^2$

11. $y = x^{-2/5} \;\Rightarrow\; y' = -\frac{2}{5}x^{(-2/5)-1} = -\frac{2}{5}x^{-7/5} = -\frac{2}{5x^{7/5}}$

13. $A(s) = -\frac{12}{s^5} = -12s^{-5} \;\Rightarrow\; A'(s) = -12(-5s^{-6}) = 60s^{-6}$ or $60/s^6$

15. $R(a) = (3a + 1)^2 = 9a^2 + 6a + 1 \;\Rightarrow\; R'(a) = 9(2a) + 6(1) + 0 = 18a + 6$

17. $S(p) = \sqrt{p} - p = p^{1/2} - p \;\Rightarrow\; S'(p) = \frac{1}{2}p^{-1/2} - 1$ or $\frac{1}{2\sqrt{p}} - 1$

19. $y = 3e^x + \frac{4}{\sqrt[3]{x}} = 3e^x + 4x^{-1/3} \;\Rightarrow\; y' = 3(e^x) + 4(-\frac{1}{3})x^{-4/3} = 3e^x - \frac{4}{3}x^{-4/3}$

21. $h(u) = Au^3 + Bu^2 + Cu \;\Rightarrow\; h'(u) = A(3u^2) + B(2u) + C(1) = 3Au^2 + 2Bu + C$

23. $y = \frac{x^2 + 4x + 3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2} \;\Rightarrow$

$y' = \frac{3}{2}x^{1/2} + 4(\frac{1}{2})x^{-1/2} + 3(-\frac{1}{2})x^{-3/2} = \frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2x\sqrt{x}}$ [note that $x^{3/2} = x^{2/2} \cdot x^{1/2} = x\sqrt{x}$]

The last expression can be written as $\frac{3x^2}{2x\sqrt{x}} + \frac{4x}{2x\sqrt{x}} - \frac{3}{2x\sqrt{x}} = \frac{3x^2 + 4x - 3}{2x\sqrt{x}}$.

25. $j(x) = x^{2.4} + e^{2.4} \;\Rightarrow\; j'(x) = 2.4x^{1.4} + 0 = 2.4x^{1.4}$

27. We first expand using the Binomial Theorem (see Reference Page 1).

$H(x) = (x + x^{-1})^3 = x^3 + 3x^2x^{-1} + 3x(x^{-1})^2 + (x^{-1})^3 = x^3 + 3x + 3x^{-1} + x^{-3} \;\Rightarrow$

$H'(x) = 3x^2 + 3 + 3(-1x^{-2}) + (-3x^{-4}) = 3x^2 + 3 - 3x^{-2} - 3x^{-4}$

61. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \lim_{h\to 0} \dfrac{x-(x+h)}{hx(x+h)} = \lim_{h\to 0} \dfrac{-h}{hx(x+h)} = \lim_{h\to 0} \dfrac{-1}{x(x+h)} = -\dfrac{1}{x^2}$

63. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \Rightarrow 2a = 2 \Rightarrow a = 1$.

$P'(2) = 3 \Rightarrow 2(1)(2) + b = 3 \Rightarrow 4 + b = 3 \Rightarrow b = -1$.

$P(2) = 5 \Rightarrow 1(2)^2 + (-1)(2) + c = 5 \Rightarrow 2 + c = 5 \Rightarrow c = 3$. So $P(x) = x^2 - x + 3$.

65. $y = f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f, so $f(-2) = 6 \Rightarrow$ $-8a + 4b - 2c + d = 6$ **(1)**. The point $(2, 0)$ is on f, so $f(2) = 0 \Rightarrow 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \Rightarrow 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \Rightarrow$ $12a + 4b + c = 0$ **(4)**. Subtracting equation (3) from (4) gives $8b = 0 \Rightarrow b = 0$. Adding (1) and (2) gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From (3) we have $c = -12a$, so (2) becomes $8a + 4(0) + 2(-12a) + 3 = 0 \Rightarrow 3 = 16a \Rightarrow$ $a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

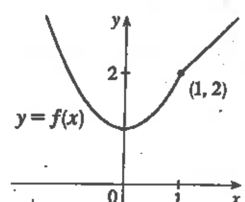
67. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$

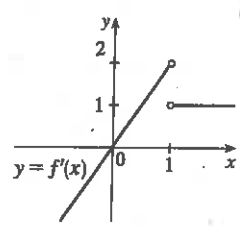
Calculate the left- and right-hand derivatives as defined in Exercise 2.8.56:

$f'_{-}(1) = \lim_{h\to 0^-} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \dfrac{[(1+h)^2+1]-(1+1)}{h} = \lim_{h\to 0^-} \dfrac{h^2+2h}{h} = \lim_{h\to 0^-}(h+2) = 2$ and

$f'_{+}(1) = \lim_{h\to 0^+} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \dfrac{[(1+h)+1]-(1+1)}{h} = \lim_{h\to 0^+} \dfrac{h}{h} = \lim_{h\to 0^+} 1 = 1$.

Since the left and right limits are different, $\lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$ does not exist, that is, $f'(1)$ does not exist. Therefore, f is not differentiable at 1.





69. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \Leftrightarrow |x| < 3 \Leftrightarrow -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \leq -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \geq 3 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

To show that $f'(3)$ does not exist we investigate $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 2.8.56.

$f'_{-}(3) = \lim_{h\to 0^-} \dfrac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \dfrac{[-(3+h)^2+9]-0}{h} = \lim_{h\to 0^-}(-6-h) = -6$ and

$f'_{+}(3) = \lim_{h\to 0^+} \dfrac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+} \dfrac{[(3+h)^2-9]-0}{h} = \lim_{h\to 0^+} \dfrac{6h+h^2}{h} = \lim_{h\to 0^+}(6+h) = 6$.

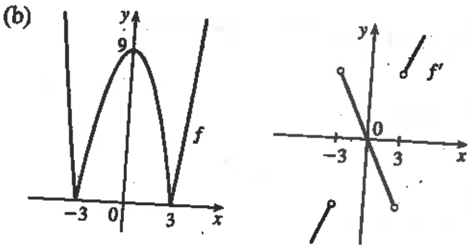
Since the left and right limits are different,

$\lim\limits_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$ does not exist, that is, $f'(3)$

does not exist. Similarly, $f'(-3)$ does not exist.

Therefore, f is not differentiable at 3 or at -3.

(b)



71. Substituting $x = 1$ and $y = 1$ into $y = ax^2 + bx$ gives us $a + b = 1$ **(1)**. The slope of the tangent line $y = 3x - 2$ is 3 and the slope of the tangent to the parabola at (x, y) is $y' = 2ax + b$. At $x = 1$, $y' = 3 \;\Rightarrow\; 3 = 2a + b$ **(2)**. Subtracting **(1)** from **(2)** gives us $2 = a$ and it follows that $b = -1$. The parabola has equation $y = 2x^2 - x$.

73. $y = f(x) = ax^2 \;\Rightarrow\; f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \;\Leftrightarrow\; y = -2x + b$, is seen to be -2, so we must have $4a = -2 \;\Leftrightarrow\; a = -\frac{1}{2}$. So when $x = 2$, the point in question has y-coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \;\Leftrightarrow\; b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

75. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.

77. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative, $f'(1) = \lim\limits_{x\to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x\to 1} \dfrac{x^{1000} - 1}{x - 1}$.
But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim\limits_{x\to 1} \dfrac{x^{1000} - 1}{x - 1} = 1000$.

Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

$$\lim_{x\to 1} \frac{x^{1000} - 1}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1} = \lim_{x\to 1} (x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$$
$$= \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{1000 \text{ ones}} = 1000, \text{ as above.}$$

79. $y = x^2 \;\Rightarrow\; y' = 2x$, so the slope of a tangent line at the point (a, a^2) is $y' = 2a$ and the slope of a normal line is $-1/(2a)$, for $a \neq 0$. The slope of the normal line through the points (a, a^2) and $(0, c)$ is $\dfrac{a^2 - c}{a - 0}$, so $\dfrac{a^2 - c}{a} = -\dfrac{1}{2a} \;\Rightarrow$ $a^2 - c = -\frac{1}{2} \;\Rightarrow\; a^2 = c - \frac{1}{2}$. The last equation has two solutions if $c > \frac{1}{2}$, one solution if $c = \frac{1}{2}$, and no solution if $c < \frac{1}{2}$. Since the y-axis is normal to $y = x^2$ regardless of the value of c (this is the case for $a = 0$), we have three normal lines if $c > \frac{1}{2}$ and one normal line if $c \leq \frac{1}{2}$.