(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

49. $T'(5)$ is the rate at which the temperature is changing at 5:00 PM. To estimate the value of $T'(5)$, we will average the difference quotients obtained using the times $t = 3$ and $t = 7$. Let $A = \dfrac{T(3) - T(5)}{3 - 5} = \dfrac{32 - 31}{-2} = -\dfrac{1}{2}$ and

$$B = \frac{T(7) - T(5)}{7 - 5} = \frac{27 - 31}{2} = -2. \text{ Then } T'(5) = \lim_{t \to 5} \frac{T(t) - T(5)}{t - 5} \approx \frac{A + B}{2} = \frac{-\frac{1}{2} - 2}{2} = -1.25\,^\circ\text{C/h}.$$

51. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\text{mg/L})/^\circ\text{C}$.

(b) For $T = 16^\circ\text{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \dfrac{6 - 14}{32 - 0} = -\dfrac{8}{32} = -0.25\ (\text{mg/L})/^\circ\text{C}$. This means that as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of $0.25\ (\text{mg/L})/^\circ\text{C}$.

53. Since $f(x) = x\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h\sin(1/h) - 0}{h} = \lim\limits_{h \to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 4 in Section 2.2.)

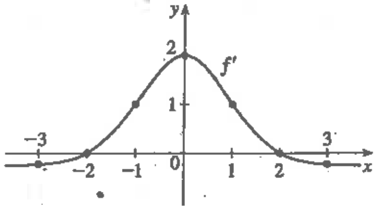
2.8 The Derivative as a Function

1. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

(a) $f'(-3) \approx -0.2$

(b) $f'(-2) \approx 0$ (c) $f'(-1) \approx 1$ (d) $f'(0) \approx 2$

(e) $f'(1) \approx 1$ (f) $f'(2) \approx 0$ (g) $f'(3) \approx -0.2$



3. (a)$'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

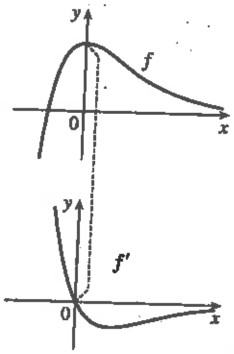
(b)$'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$'$ = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)$' =$ III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 4–11: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f—there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

5.



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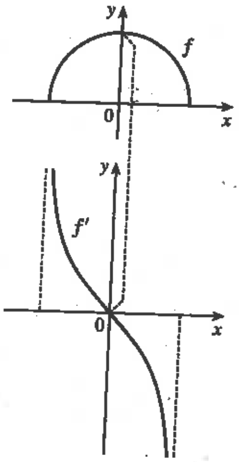

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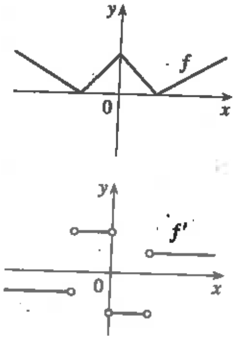

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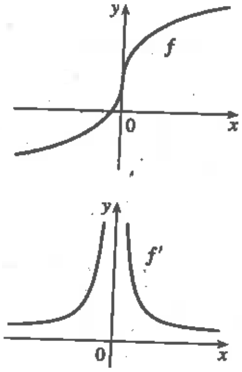
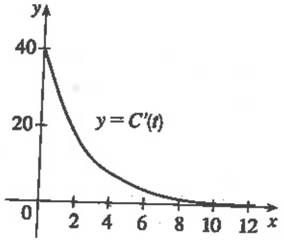
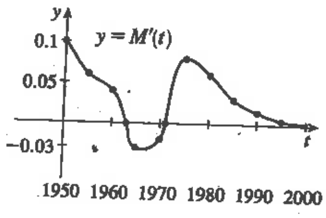


13. (a) $C'(t)$ is the instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours.

(b) The graph of $C'(t)$ tells us that the rate of change of percentage of full capacity is decreasing and approaching 0.



15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.



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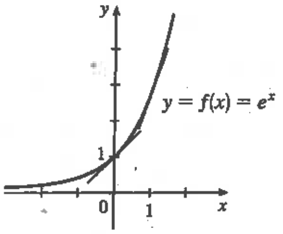
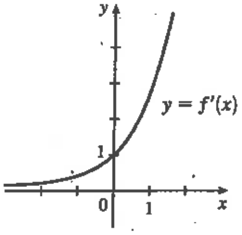
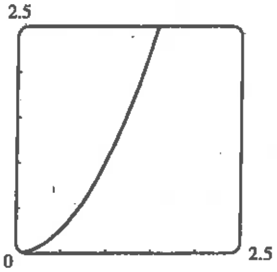




The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7. As x decreases, the slope gets closer to 0. Since the graphs are so similar, we might guess that $f'(x) = e^x$.

19. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.



(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.

(d) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{\left(x^2 + 2hx + h^2\right) - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x$$

21. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[\frac{1}{2}(x+h) - \frac{1}{3}\right] - \left(\frac{1}{2}x - \frac{1}{3}\right)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}x + \frac{1}{2}h - \frac{1}{3} - \frac{1}{2}x + \frac{1}{3}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{2}h}{h} = \lim_{h\to 0} \tfrac{1}{2} = \tfrac{1}{2}$$

Domain of f = domain of $f' = \mathbb{R}$.

23. $$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{\left[5(t+h) - 9(t+h)^2\right] - \left(5t - 9t^2\right)}{h}$$
$$= \lim_{h\to 0} \frac{5t + 5h - 9(t^2 + 2th + h^2) - 5t + 9t^2}{h} = \lim_{h\to 0} \frac{5t + 5h - 9t^2 - 18th - 9h^2 - 5t + 9t^2}{h}$$
$$= \lim_{h\to 0} \frac{5h - 18th - 9h^2}{h} = \lim_{h\to 0} \frac{h(5 - 18t - 9h)}{h} = \lim_{h\to 0}(5 - 18t - 9h) = 5 - 18t$$

Domain of f = domain of $f' = \mathbb{R}$.

25. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left[(x+h)^3-3(x+h)+5\right]-(x^3-3x+5)}{h}$$
$$= \lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3-3x-3h+5)-(x^3-3x+5)}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h}$$
$$= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2-3)}{h} = \lim_{h\to 0}\left(3x^2+3xh+h^2-3\right) = 3x^2-3$$

Domain of f = domain of $f' = \mathbb{R}$.

27. $$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h}\left[\frac{\sqrt{9-(x+h)}+\sqrt{9-x}}{\sqrt{9-(x+h)}+\sqrt{9-x}}\right]$$
$$= \lim_{h\to 0}\frac{[9-(x+h)]-(9-x)}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]} = \lim_{h\to 0}\frac{-h}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]}$$
$$= \lim_{h\to 0}\frac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}} = \frac{-1}{2\sqrt{9-x}}$$

Domain of $g = (-\infty, 9]$, domain of $g' = (-\infty, 9)$.

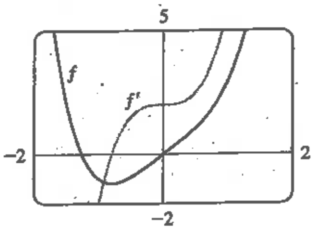
29. $$G'(t) = \lim_{h\to 0}\frac{G(t+h)-G(t)}{h} = \lim_{h\to 0}\frac{\dfrac{1-2(t+h)}{3+(t+h)}-\dfrac{1-2t}{3+t}}{h}$$
$$= \lim_{h\to 0}\frac{\dfrac{[1-2(t+h)](3+t)-[3+(t+h)](1-2t)}{[3+(t+h)](3+t)}}{h}$$
$$= \lim_{h\to 0}\frac{3+t-6t-2t^2-6h-2ht-(3-6t+t-2t^2+h-2ht)}{h[3+(t+h)](3+t)} = \lim_{h\to 0}\frac{-6h-h}{h(3+t+h)(3+t)}$$
$$= \lim_{h\to 0}\frac{-7h}{h(3+t+h)(3+t)} = \lim_{h\to 0}\frac{-7}{(3+t+h)(3+t)} = \frac{-7}{(3+t)^2}$$

Domain of G = domain of $G' = (-\infty, -3)\cup(-3, \infty)$.

31. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^4-x^4}{h} = \lim_{h\to 0}\frac{(x^4+4x^3h+6x^2h^2+4xh^3+h^4)-x^4}{h}$$
$$= \lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0}\left(4x^3+6x^2h+4xh^2+h^3\right) = 4x^3$$

Domain of f = domain of $f' = \mathbb{R}$.

33. (a) $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^4+2(x+h)]-(x^4+2x)}{h}$$
$$= \lim_{h\to 0}\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4+2x+2h-x^4-2x}{h}$$
$$= \lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4+2h}{h} = \lim_{h\to 0}\frac{h(4x^3+6x^2h+4xh^2+h^3+2)}{h}$$
$$= \lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3+2) = 4x^3+2$$



(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope.

35. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find $U'(t)$, we use $\lim\limits_{h\to 0} \dfrac{U(t+h)-U(t)}{h} \approx \dfrac{U(t+h)-U(t)}{h}$ for small values of h.

For 1995: $U'(1995) \approx \dfrac{U(1996)-U(1995)}{1996-1995} = \dfrac{8.0-8.1}{1} = -0.10$

For 1996: We estimate $U'(1996)$ by using $h=-1$ and $h=1$, and then average the two results to obtain a final estimate.

$$h=-1 \;\Rightarrow\; U'(1996) \approx \frac{U(1995)-U(1996)}{1995-1996} = \frac{8.1-8.0}{-1} = -0.10;$$

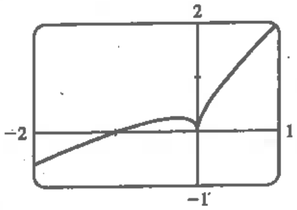
$$h=1 \;\Rightarrow\; U'(1996) \approx \frac{U(1997)-U(1996)}{1997-1996} = \frac{8.2-8.0}{1} = 0.20.$$

So we estimate that $U'(1996) \approx \frac{1}{2}[-0.10+0.20] = 0.05$.

t	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004
$U'(t)$	−0.10	0.05	−0.05	−0.75	−0.85	0.10	0.15	−0.35	−0.45	−0.60

37. f is not differentiable at $x=-4$, because the graph has a corner there, and at $x=0$, because there is a discontinuity there.

39. f is not differentiable at $x=-1$, because the graph has a vertical tangent there, and at $x=4$, because the graph has a corner there.



41. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x=-1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x=0$.

43. $a=f$, $b=f'$, $c=f''$. We can see this because where a has a horizontal tangent, $b=0$, and where b has a horizontal tangent, $c=0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

45. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a=0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b'=a$. We conclude that c is the graph of the position function.

47.
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[3(x+h)^2+2(x+h)+1]-(3x^2+2x+1)}{h}$$
$$= \lim_{h\to 0}\frac{(3x^2+6xh+3h^2+2x+2h+1)-(3x^2+2x+1)}{h} = \lim_{h\to 0}\frac{6xh+3h^2+2h}{h}$$
$$= \lim_{h\to 0}\frac{h(6x+3h+2)}{h} = \lim_{h\to 0}(6x+3h+2) = 6x+2$$