corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that $\lim\limits_{t\to 0^+} f(1/t) = L = \lim\limits_{x\to\infty} f(x)$.

Now suppose that $\lim\limits_{x\to -\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \Leftrightarrow 1/N < 1/x < 0 \Leftrightarrow 1/N < t < 0$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\lim\limits_{t\to 0^-} f(1/t) = L = \lim\limits_{x\to -\infty} f(x)$.

2.7 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim\limits_{x\to 3} \dfrac{f(x) - f(3)}{x - 3}$.

3. (a) (i) Using Definition 1 with $f(x) = 4x - x^2$ and $P(1, 3)$,

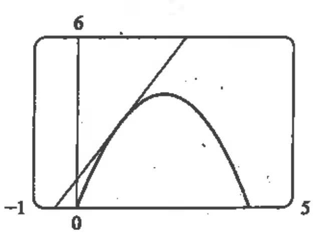
$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 1} \frac{(4x - x^2) - 3}{x - 1} = \lim_{x\to 1} \frac{-(x^2 - 4x + 3)}{x - 1} = \lim_{x\to 1} \frac{-(x - 1)(x - 3)}{x - 1}$$
$$= \lim_{x\to 1}(3 - x) = 3 - 1 = 2$$

(ii) Using Equation 2 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h\to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h\to 0} \frac{[4(1 + h) - (1 + h)^2] - 3}{h}$$
$$= \lim_{h\to 0} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h\to 0} \frac{-h^2 + 2h}{h} = \lim_{h\to 0} \frac{h(-h + 2)}{h} = \lim_{h\to 0}(-h + 2) = 2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \Rightarrow y - f(1) = f'(1)(x - 1) \Rightarrow y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(c)



The graph of $y = 2x + 1$ is tangent to the graph of $y = 4x - x^2$ at the point $(1, 3)$. Now zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistiguishable.

5. Using (1) with $f(x) = 4x - 3x^2$ and $P(2, -4)$ [we could also use (2)],

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 2} \frac{(4x - 3x^2) - (-4)}{x - 2} = \lim_{x\to 2} \frac{-3x^2 + 4x + 4}{x - 2}$$
$$= \lim_{x\to 2} \frac{(-3x - 2)(x - 2)}{x - 2} = \lim_{x\to 2}(-3x - 2) = -3(2) - 2 = -8$$

Tangent line: $y - (-4) = -8(x - 2) \Leftrightarrow y + 4 = -8x + 16 \Leftrightarrow y = -8x + 12$.

7. Using (1), $m = \lim_{x \to 1} \dfrac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x \to 1} \dfrac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \dfrac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \dfrac{1}{\sqrt{x} + 1} = \dfrac{1}{2}$.

Tangent line: $y - 1 = \frac{1}{2}(x - 1) \iff y = \frac{1}{2}x + \frac{1}{2}$

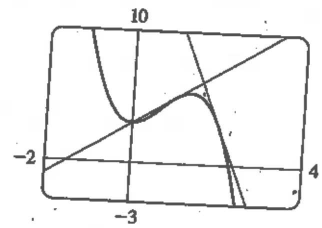
9. (a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$,

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{3 + 4(a + h)^2 - 2(a + h)^3 - (3 + 4a^2 - 2a^3)}{h} \\ &= \lim_{h \to 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h} \\ &= \lim_{h \to 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h} \\ &= \lim_{h \to 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h} = \lim_{h \to 0} \frac{h(8a + 4h - 6a^2 - 6ah - 2h^2)}{h} \\ &= \lim_{h \to 0} (8a + 4h - 6a^2 - 6ah - 2h^2) = 8a - 6a^2 \end{aligned}$$

(b) At $(1, 5)$: $m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is $y - 5 = 2(x - 1) \iff y = 2x + 3$.

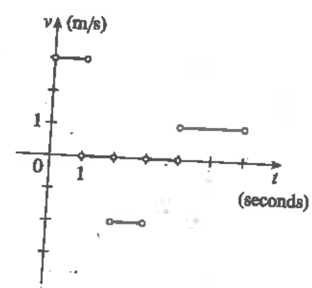
At $(2, 3)$: $m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent line is $y - 3 = -8(x - 2) \iff y = -8x + 19$.

(c)



11. (a) The particle is moving to the right when s is increasing; that is, on the intervals $(0, 1)$ and $(4, 6)$. The particle is moving to the left when s is decreasing; that is, on the interval $(2, 3)$. The particle is standing still when s is constant; that is, on the intervals $(1, 2)$ and $(3, 4)$.

(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval $(0, 1)$, the slope is $\dfrac{3 - 0}{1 - 0} = 3$. On the interval $(2, 3)$, the slope is $\dfrac{1 - 3}{3 - 2} = -2$. On the interval $(4, 6)$, the slope is $\dfrac{3 - 1}{6 - 4} = 1$.



13. Let $s(t) = 10t - 4.9t^2$.

$$\begin{aligned} v(2) &= \lim_{h \to 0} \frac{s(2 + h) - s(2)}{h} = \lim_{h \to 0} \frac{[10(2 + h) - 4.9(2 + h)^2] - 0.4}{h} = \lim_{h \to 0} \frac{20 + 10h - 4.9(4 + 4h + h^2) - 0.4}{h} \\ &= \lim_{h \to 0} \frac{-4.9h^2 - 9.6h}{h} = \lim_{h \to 0} (-4.9h - 9.6) = -9.6 \end{aligned}$$

Thus, the instantaneous velocity when $t = 2$ is -9.6 m/s.

15. $v(a) = \lim_{h\to 0} \dfrac{s(a+h)-s(a)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \dfrac{\dfrac{a^2-(a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \dfrac{a^2-(a^2+2ah+h^2)}{ha^2(a+h)^2}$

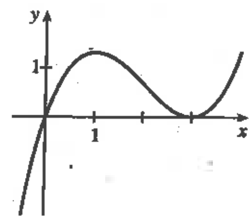
$= \lim_{h\to 0} \dfrac{-(2ah+h^2)}{ha^2(a+h)^2} = \lim_{h\to 0} \dfrac{-h(2a+h)}{ha^2(a+h)^2} = \lim_{h\to 0} \dfrac{-(2a+h)}{a^2(a+h)^2} = \dfrac{-2a}{a^2\cdot a^2} = \dfrac{-2}{a^3}$ m/s

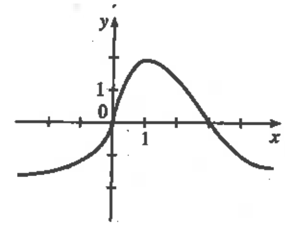
So $v(1) = \dfrac{-2}{1^3} = -2$ m/s, $v(2) = \dfrac{-2}{2^3} = -\dfrac{1}{4}$ m/s, and $v(3) = \dfrac{-2}{3^3} = -\dfrac{2}{27}$ m/s.

17. $g'(0)$ is the only negative value. The slope at $x=4$ is smaller than the slope at $x=2$ and both are smaller than the slope at $x=-2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

19. For the tangent line $y = 4x-5$: when $x=2$, $y = 4(2)-5 = 3$ and its slope is 4 (the coefficient of x). At the point of tangency, these values are shared with the curve $y=f(x)$; that is, $f(2)=3$ and $f'(2)=4$.

21. We begin by drawing a curve through the origin with a slope of 3 to satisfy $f(0)=0$ and $f'(0)=3$. Since $f'(1)=0$, we will round off our figure so that there is a horizontal tangent directly over $x=1$. Last, we make sure that the curve has a slope of -1 as we pass over $x=2$. Two of the many possibilities are shown.





23. Using (4) with $f(x) = 3x^2 - x^3$ and $a=1$,

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{[3(1+h)^2-(1+h)^3]-2}{h}$$

$$= \lim_{h\to 0} \frac{(3+6h+3h^2)-(1+3h+3h^2+h^3)-2}{h} = \lim_{h\to 0} \frac{3h-h^3}{h} = \lim_{h\to 0} \frac{h(3-h^2)}{h}$$

$$= \lim_{h\to 0}(3-h^2) = 3-0 = 3$$

Tangent line: $y-2 = 3(x-1) \;\Leftrightarrow\; y-2 = 3x-3 \;\Leftrightarrow\; y = 3x-1$

25. (a) Using (4) with $F(x) = 5x/(1+x^2)$ and the point $(2,2)$, we have

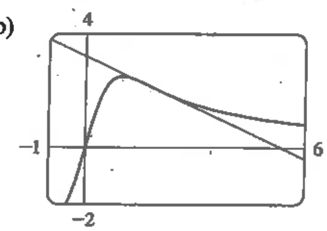
$$F'(2) = \lim_{h\to 0} \frac{F(2+h)-F(2)}{h} = \lim_{h\to 0} \frac{\dfrac{5(2+h)}{1+(2+h)^2} - 2}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{5h+10}{h^2+4h+5} - 2}{h} = \lim_{h\to 0} \frac{\dfrac{5h+10-2(h^2+4h+5)}{h^2+4h+5}}{h}$$

$$= \lim_{h\to 0} \frac{-2h^2-3h}{h(h^2+4h+5)} = \lim_{h\to 0} \frac{h(-2h-3)}{h(h^2+4h+5)} = \lim_{h\to 0} \frac{-2h-3}{h^2+4h+5} = \frac{-3}{5}$$

So an equation of the tangent line at $(2,2)$ is $y-2 = -\frac{3}{5}(x-2)$ or $y = -\frac{3}{5}x + \frac{16}{5}$.

(b)



27. Use (4) with $f(x) = 3x^2 - 4x + 1$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{[3(a+h)^2 - 4(a+h) + 1] - (3a^2 - 4a + 1)]}{h}$$

$$= \lim_{h\to 0} \frac{3a^2 + 6ah + 3h^2 - 4a - 4h + 1 - 3a^2 + 4a - 1}{h} = \lim_{h\to 0} \frac{6ah + 3h^2 - 4h}{h}$$

$$= \lim_{h\to 0} \frac{h(6a + 3h - 4)}{h} = \lim_{h\to 0} (6a + 3h - 4) = 6a - 4$$

29. Use (4) with $f(t) = (2t+1)/(t+3)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{2(a+h)+1}{(a+h)+3} - \dfrac{2a+1}{a+3}}{h} = \lim_{h\to 0} \frac{(2a+2h+1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h\to 0} \frac{(2a^2 + 6a + 2ah + 6h + a + 3) - (2a^2 + 2ah + 6a + a + h + 3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h\to 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h\to 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2}$$

31. Use (4) with $f(x) = \sqrt{1-2x}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h} \cdot \frac{\sqrt{1-2(a+h)} + \sqrt{1-2a}}{\sqrt{1-2(a+h)} + \sqrt{1-2a}} = \lim_{h\to 0} \frac{\left(\sqrt{1-2(a+h)}\right)^2 - \left(\sqrt{1-2a}\right)^2}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)}$$

$$= \lim_{h\to 0} \frac{(1 - 2a - 2h) - (1 - 2a)}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} = \lim_{h\to 0} \frac{-2h}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)}$$

$$= \lim_{h\to 0} \frac{-2}{\sqrt{1-2(a+h)} + \sqrt{1-2a}} = \frac{-2}{\sqrt{1-2a} + \sqrt{1-2a}} = \frac{-2}{2\sqrt{1-2a}} = \frac{-1}{\sqrt{1-2a}}$$

Note that the answers to Exercises 33–38 are not unique.

33. By (4), $\displaystyle\lim_{h\to 0} \frac{(1+h)^{10} - 1}{h} = f'(1)$, where $f(x) = x^{10}$ and $a = 1$.

Or: By (4), $\displaystyle\lim_{h\to 0} \frac{(1+h)^{10} - 1}{h} = f'(0)$, where $f(x) = (1+x)^{10}$ and $a = 0$.

35. By Equation 5, $\displaystyle\lim_{x\to 5} \frac{2^x - 32}{x - 5} = f'(5)$, where $f(x) = 2^x$ and $a = 5$.

37. By (4), $\displaystyle\lim_{h\to 0} \frac{\cos(\pi + h) + 1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By (4), $\displaystyle\lim_{h\to 0} \frac{\cos(\pi + h) + 1}{h} = f'(0)$, where $f(x) = \cos(\pi + x)$ and $a = 0$.

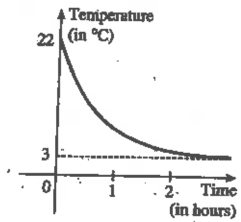
39. $v(5) = f'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h \to 0} \frac{[100 + 50(5+h) - 4.9(5+h)^2] - [100 + 50(5) - 4.9(5)^2]}{h}$

$= \lim_{h \to 0} \frac{(100 + 250 + 50h - 4.9h^2 - 49h - 122.5) - (100 + 250 - 122.5)}{h} = \lim_{h \to 0} \frac{-4.9h^2 + h}{h}$

$= \lim_{h \to 0} \frac{h(-4.9h + 1)}{h} = \lim_{h \to 0} (-4.9h + 1) = 1 \text{ m/s}$

The speed when $t = 5$ is $|1| = 1$ m/s.

41. The sketch shows the graph for a room temperature of 22° and a refrigerator temperature of 3°. The initial rate of change is greater in magnitude than the rate of change after an hour.



43. (a) (i) $[2001, 2005]$: $\frac{P(2005) - P(2001)}{2005 - 2001} = \frac{11.78 - 8.49}{4} = \frac{3.29}{4} = 0.82$

(ii) $[2003, 2005]$: $\frac{P(2005) - P(2003)}{2005 - 2003} = \frac{11.78 - 9.65}{2} = \frac{2.13}{2} \approx 1.07$

(iii) $[2005, 2007]$: $\frac{P(2007) - P(2005)}{2007 - 2005} = \frac{14.54 - 11.78}{2} = \frac{2.76}{2} = 1.38$

(b) Using the values from (ii) and (iii), we have $\frac{1.07 + 1.38}{2} \approx 1.23$ millions of passengers per year.

45. (a) (i) $\frac{\Delta C}{\Delta x} = \frac{C(105) - C(100)}{105 - 100} = \frac{6601.25 - 6500}{5} = \$20.25/\text{unit}.$

(ii) v

$\frac{\Delta C}{\Delta x} = \frac{C(101) - C(100)}{101 - 100} = \frac{6520.05 - 6500}{1} = \$20.05/\text{unit}.$

(b) $\frac{C(100+h) - C(100)}{h} = \frac{[5000 + 10(100+h) + 0.05(100+h)^2] - 6500}{h} = \frac{20h + 0.05h^2}{h}$

$= 20 + 0.05h,\ h \neq 0$

So the instantaneous rate of change is $\lim_{h \to 0} \frac{C(100+h) - C(100)}{h} = \lim_{h \to 0} (20 + 0.05h) = \$20/\text{unit}.$

47. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of kilograms of gold produced. Its units are dollars per kilogram.

(b) After 50 kilograms of gold have been produced, the rate at which the production cost is increasing is \$36/kilogram. So the cost of producing the 50th (or 51st) kilogram is about \$36.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

49. $T'(5)$ is the rate at which the temperature is changing at 5:00 PM. To estimate the value of $T'(5)$, we will average the difference quotients obtained using the times $t = 3$ and $t = 7$. Let $A = \dfrac{T(3) - T(5)}{3 - 5} = \dfrac{32 - 31}{-2} = -\dfrac{1}{2}$ and

$$B = \frac{T(7) - T(5)}{7 - 5} = \frac{27 - 31}{2} = -2. \text{ Then } T'(5) = \lim_{t \to 5} \frac{T(t) - T(5)}{t - 5} \approx \frac{A + B}{2} = \frac{-\frac{1}{2} - 2}{2} = -1.25\,^\circ\text{C/h}.$$

51. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\text{mg/L})/^\circ\text{C}$.

(b) For $T = 16^\circ\text{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So

$S'(16) \approx \dfrac{6 - 14}{32 - 0} = -\dfrac{8}{32} = -0.25\ (\text{mg/L})/^\circ\text{C}$. This means that as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of $0.25\ (\text{mg/L})/^\circ\text{C}$.

53. Since $f(x) = x\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h\sin(1/h) - 0}{h} = \lim\limits_{h \to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 4 in Section 2.2.)

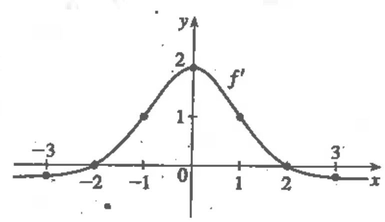
2.8 The Derivative as a Function

1. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

(a) $f'(-3) \approx -0.2$

(b) $f'(-2) \approx 0$ (c) $f'(-1) \approx 1$ (d) $f'(0) \approx 2$

(e) $f'(1) \approx 1$ (f) $f'(2) \approx 0$ (g) $f'(3) \approx -0.2$



3. (a)$' = $ II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

(b)$' = $ IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$' = $ I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.