59. (⇒) If f is continuous at a, then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h\to 0} f(a+h) = f\left(\lim_{h\to 0}(a+h)\right) = f(a).$$

(⇐) Let $\varepsilon > 0$. Since $\lim_{h\to 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \Rightarrow$ $|f(a+h) - f(a)| < \varepsilon$. So if $0 < |x-a| < \delta$, then $|f(x) - f(a)| = |f(a + (x-a)) - f(a)| < \varepsilon$.

Thus, $\lim_{x\to a} f(x) = f(a)$ and so f is continuous at a.

61. As in the previous exercise, we must show that $\lim_{h\to 0}\cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\begin{aligned}\lim_{h\to 0}\cos(a+h) &= \lim_{h\to 0}(\cos a\cos h - \sin a\sin h) = \lim_{h\to 0}(\cos a\cos h) - \lim_{h\to 0}(\sin a\sin h)\\ &= \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\cos h\right) - \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\sin h\right) = (\cos a)(1) - (\sin a)(0) = \cos a\end{aligned}$$

63. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational}\end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x\to a} f(x) \neq f(a)$. [In fact, $\lim_{x\to a} f(x)$ does not even exist.]

65. If there is such a number, it satisfies the equation $x^3 + 1 = x \Leftrightarrow x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

67. $f(x) = x^4\sin(1/x)$ is continuous on $(-\infty, 0)\cup(0,\infty)$ since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since $-1 \le \sin(1/x) \le 1$, we have $-x^4 \le x^4\sin(1/x) \le x^4$. Because $\lim_{x\to 0}(-x^4) = 0$ and $\lim_{x\to 0} x^4 = 0$, the Squeeze Theorem gives us $\lim_{x\to 0}(x^4\sin(1/x)) = 0$, which equals $f(0)$. Thus, f is continuous at 0 and, hence, on $(-\infty,\infty)$.

69. Define $u(t)$ to be the monk's distance from the monastery, as a function of time t (in hours), on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u-d)(0) = -D$ and $(u-d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u-d)(t_0) = 0 \Leftrightarrow$ $u(t_0) = d(t_0)$. So at time t_0 after 7:00 AM, the monk will be at the same place on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

1. (a) As x becomes large, the values of $f(x)$ approach 5.

(b) As x becomes large negative, the values of $f(x)$ approach 3.

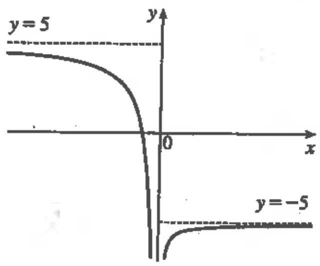
3. (a) $\lim_{x\to 2} f(x) = \infty$ (b) $\lim_{x\to -1^-} f(x) = \infty$ (c) $\lim_{x\to -1^+} f(x) = -\infty$

(d) $\lim_{x\to\infty} f(x) = 1$ (e) $\lim_{x\to -\infty} f(x) = 2$ (f) Vertical: $x = -1$, $x = 2$; Horizontal: $y = 1$, $y = 2$

5. $\lim_{x\to 0} f(x) = -\infty$,

$\lim_{x\to -\infty} f(x) = 5$,

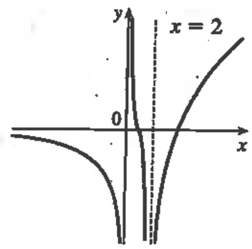
$\lim_{x\to \infty} f(x) = -5$



7. $\lim_{x\to 2} f(x) = -\infty$, $\lim_{x\to \infty} f(x) = \infty$,

$\lim_{x\to -\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$,

$\lim_{x\to 0^-} f(x) = -\infty$

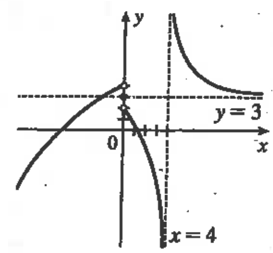


9. $f(0) = 3$, $\lim_{x\to 0^-} f(x) = 4$,

$\lim_{x\to 0^+} f(x) = 2$,

$\lim_{x\to -\infty} f(x) = -\infty$, $\lim_{x\to 4^-} f(x) = -\infty$,

$\lim_{x\to 4^+} f(x) = \infty$, $\lim_{x\to \infty} f(x) = 3$



11. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$.

It appears that $\lim_{x\to\infty} (x^2/2^x) = 0$.

13. $$\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8} = \lim_{x\to\infty} \frac{(3x^2 - x + 4)/x^2}{(2x^2 + 5x - 8)/x^2}$$ [divide both the numerator and denominator by x^2 (the highest power of x that appears in the denominator)]

$$= \frac{\lim_{x\to\infty}(3 - 1/x + 4/x^2)}{\lim_{x\to\infty}(2 + 5/x - 8/x^2)}$$ [Limit Law 5]

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}(1/x) + \lim_{x\to\infty}(4/x^2)}{\lim_{x\to\infty} 2 + \lim_{x\to\infty}(5/x) - \lim_{x\to\infty}(8/x^2)}$$ [Limit Laws 1 and 2]

$$= \frac{3 - \lim_{x\to\infty}(1/x) + 4\lim_{x\to\infty}(1/x^2)}{2 + 5\lim_{x\to\infty}(1/x) - 8\lim_{x\to\infty}(1/x^2)}$$ [Limit Laws 7 and 3]

$$= \frac{3 - 0 + 4(0)}{2 + 5(0) - 8(0)}$$ [Theorem 5 of Section 2.5]

$$= \frac{3}{2}$$

15. $$\lim_{x\to\infty} \frac{1}{2x+3} = \lim_{x\to\infty} \frac{1/x}{(2x+3)/x} = \frac{\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty}(2 + 3/x)} = \frac{\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty} 2 + 3\lim_{x\to\infty}(1/x)} = \frac{0}{2 + 3(0)} = \frac{0}{2} = 0$$

17. $$\lim_{x\to -\infty} \frac{1 - x - x^2}{2x^2 - 7} = \lim_{x\to -\infty} \frac{(1 - x - x^2)/x^2}{(2x^2 - 7)/x^2} = \frac{\lim_{x\to -\infty}(1/x^2 - 1/x - 1)}{\lim_{x\to -\infty}(2 - 7/x^2)}$$

$$= \frac{\lim_{x\to -\infty}(1/x^2) - \lim_{x\to -\infty}(1/x) - \lim_{x\to -\infty} 1}{\lim_{x\to -\infty} 2 - 7\lim_{x\to -\infty}(1/x^2)} = \frac{0 - 0 - 1}{2 - 7(0)} = -\frac{1}{2}$$

19. $\lim\limits_{t\to\infty}\dfrac{\sqrt{t}+t^2}{2t-t^2}=\lim\limits_{t\to\infty}\dfrac{(\sqrt{t}+t^2)/t^2}{(2t-t^2)/t^2}=\lim\limits_{t\to\infty}\dfrac{1/t^{3/2}+1}{2/t-1}=\dfrac{0+1}{0-1}=-1$

21. $\lim\limits_{x\to\infty}\dfrac{(2x^2+1)^2}{(x-1)^2(x^2+x)}=\lim\limits_{x\to\infty}\dfrac{(2x^2+1)^2/x^4}{[(x-1)^2(x^2+x)]/x^4}=\lim\limits_{x\to\infty}\dfrac{[(2x^2+1)/x^2]^2}{[(x^2-2x+1)/x^2][(x^2+x)/x^2]}$

$=\lim\limits_{x\to\infty}\dfrac{(2+1/x^2)^2}{(1-2/x+1/x^2)(1+1/x)}=\dfrac{(2+0)^2}{(1-0+0)(1+0)}=4$

23. $\lim\limits_{x\to\infty}\dfrac{\sqrt{9x^6-x}}{x^3+1}=\lim\limits_{x\to\infty}\dfrac{\sqrt{9x^6-x}/x^3}{(x^3+1)/x^3}=\dfrac{\lim\limits_{x\to\infty}\sqrt{(9x^6-x)/x^6}}{\lim\limits_{x\to\infty}(1+1/x^3)}$ [since $x^3=\sqrt{x^6}$ for $x>0$]

$=\dfrac{\lim\limits_{x\to\infty}\sqrt{9-1/x^5}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}(1/x^3)}=\dfrac{\sqrt{\lim\limits_{x\to\infty}9-\lim\limits_{x\to\infty}(1/x^5)}}{1+0}=\sqrt{9-0}=3$

25. $\lim\limits_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)=\lim\limits_{x\to\infty}\dfrac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x}=\lim\limits_{x\to\infty}\dfrac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}$

$=\lim\limits_{x\to\infty}\dfrac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x}=\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{9x^2+x}+3x}\cdot\dfrac{1/x}{1/x}$

$=\lim\limits_{x\to\infty}\dfrac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x}=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{9+1/x}+3}=\dfrac{1}{\sqrt{9}+3}=\dfrac{1}{3+3}=\dfrac{1}{6}$

27. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)=\lim\limits_{x\to\infty}\dfrac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$

$=\lim\limits_{x\to\infty}\dfrac{(x^2+ax)-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\lim\limits_{x\to\infty}\dfrac{[(a-b)x]/x}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)/\sqrt{x^2}}$

$=\lim\limits_{x\to\infty}\dfrac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}}=\dfrac{a-b}{\sqrt{1+0}+\sqrt{1+0}}=\dfrac{a-b}{2}$

29. $\lim\limits_{x\to\infty}\dfrac{x^4-3x^2+x}{x^3-x+2}=\lim\limits_{x\to\infty}\dfrac{(x^4-3x^2+x)/x^3}{(x^3-x+2)/x^3}$ [divide by the highest power of x in the denominator] $=\lim\limits_{x\to\infty}\dfrac{x-3/x+1/x^2}{1-1/x^2+2/x^3}=\infty$

since the numerator increases without bound and the denominator approaches 1 as $x\to\infty$.

31. $\lim\limits_{x\to-\infty}(x^4+x^5)=\lim\limits_{x\to-\infty}x^5(\frac{1}{x}+1)$ [factor out the largest power of x] $=-\infty$ because $x^5\to-\infty$ and $1/x+1\to1$ as $x\to-\infty$.

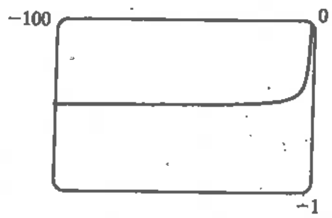
Or: $\lim\limits_{x\to-\infty}(x^4+x^5)=\lim\limits_{x\to-\infty}x^4(1+x)=-\infty$.

33. Let $t=e^x$. As $x\to\infty$, $t\to\infty$. $\lim\limits_{x\to\infty}\arctan(e^x)=\lim\limits_{t\to\infty}\arctan t=\frac{\pi}{2}$ by (3).

35. $\lim\limits_{x\to\infty}\dfrac{1-e^x}{1+2e^x}=\lim\limits_{x\to\infty}\dfrac{(1-e^x)/e^x}{(1+2e^x)/e^x}=\lim\limits_{x\to\infty}\dfrac{1/e^x-1}{1/e^x+2}=\dfrac{0-1}{0+2}=-\dfrac{1}{2}$

37. Since $-1\le\cos x\le1$ and $e^{-2x}>0$, we have $-e^{-2x}\le e^{-2x}\cos x\le e^{-2x}$. We know that $\lim\limits_{x\to\infty}(-e^{-2x})=0$ and $\lim\limits_{x\to\infty}\left(e^{-2x}\right)=0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty}(e^{-2x}\cos x)=0$.

39. (a)



From the graph of $f(x) = \sqrt{x^2 + x + 1} + x$, we estimate the value of $\lim_{x \to -\infty} f(x)$ to be -0.5.

(b)

x	$f(x)$
−10,000	−0.4999625
−100,000	−0.4999962
−1,000,000	−0.4999996

From the table, we estimate the limit to be -0.5.

(c) $\lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right) = \lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right)\left[\dfrac{\sqrt{x^2 + x + 1} - x}{\sqrt{x^2 + x + 1} - x}\right] = \lim_{x \to -\infty} \dfrac{(x^2 + x + 1) - x^2}{\sqrt{x^2 + x + 1} - x}$

$$= \lim_{x \to -\infty} \frac{(x+1)(1/x)}{\left(\sqrt{x^2 + x + 1} - x\right)(1/x)} = \lim_{x \to -\infty} \frac{1 + (1/x)}{-\sqrt{1 + (1/x) + (1/x^2)} - 1}$$

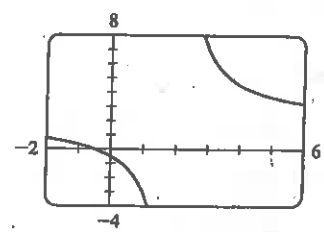
$$= \frac{1 + 0}{-\sqrt{1 + 0 + 0} - 1} = -\frac{1}{2}$$

Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x, with $x < 0$, we get $\dfrac{1}{x}\sqrt{x^2 + x + 1} = -\dfrac{1}{\sqrt{x^2}}\sqrt{x^2 + x + 1} = -\sqrt{1 + (1/x) + (1/x^2)}$.

41. $\displaystyle \lim_{x \to \infty} \frac{2x + 1}{x - 2} = \lim_{x \to \infty} \frac{\frac{2x + 1}{x}}{\frac{x - 2}{x}} = \lim_{x \to \infty} \frac{2 + \frac{1}{x}}{1 - \frac{2}{x}} = \frac{\lim_{x \to \infty}\left(2 + \frac{1}{x}\right)}{\lim_{x \to \infty}\left(1 - \frac{2}{x}\right)} = \frac{\lim_{x \to \infty} 2 + \lim_{x \to \infty} \frac{1}{x}}{\lim_{x \to \infty} 1 - \lim_{x \to \infty} \frac{2}{x}}$

$\displaystyle = \frac{2 + 0}{1 - 0} = 2$, so $y = 2$ is a horizontal asymptote.

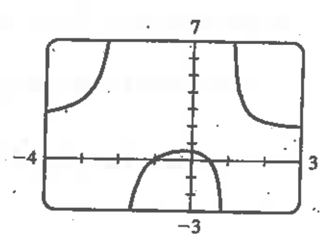
The denominator $x - 2$ is zero when $x = 2$ and the numerator is not zero, so we investigate $y = f(x) = \dfrac{2x + 1}{x - 2}$ as x approaches 2. $\lim_{x \to 2^-} f(x) = -\infty$ because as $x \to 2^-$ the numerator is positive and the denominator approaches 0 through negative values. Similarly, $\lim_{x \to 2^+} f(x) = \infty$. Thus, $x = 2$ is a vertical asymptote. The graph confirms our work.



43. $\displaystyle \lim_{x \to \infty} \frac{2x^2 + x - 1}{x^2 + x - 2} = \lim_{x \to \infty} \frac{\frac{2x^2 + x - 1}{x^2}}{\frac{x^2 + x - 2}{x^2}} = \lim_{x \to \infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}} = \frac{\lim_{x \to \infty}\left(2 + \frac{1}{x} - \frac{1}{x^2}\right)}{\lim_{x \to \infty}\left(1 + \frac{1}{x} - \frac{2}{x^2}\right)}$

$\displaystyle = \frac{\lim_{x \to \infty} 2 + \lim_{x \to \infty} \frac{1}{x} - \lim_{x \to \infty} \frac{1}{x^2}}{\lim_{x \to \infty} 1 + \lim_{x \to \infty} \frac{1}{x} - 2 \lim_{x \to \infty} \frac{1}{x^2}} = \frac{2 + 0 - 0}{1 + 0 - 2(0)} = 2$, so $y = 2$ is a horizontal asymptote.

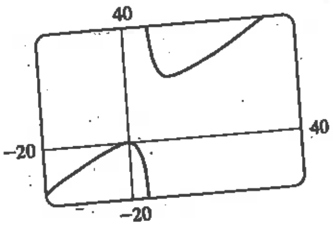
$y = f(x) = \dfrac{2x^2 + x - 1}{x^2 + x - 2} = \dfrac{(2x - 1)(x + 1)}{(x + 2)(x - 1)}$, so $\lim_{x \to -2^-} f(x) = \infty$, $\lim_{x \to -2^+} f(x) = -\infty$, $\lim_{x \to 1^-} f(x) = -\infty$, and $\lim_{x \to 1^+} f(x) = \infty$. Thus, $x = -2$ and $x = 1$ are vertical asymptotes. The graph confirms our work.



45. $y = f(x) = \dfrac{x^3 - x}{x^2 - 6x + 5} = \dfrac{x(x^2-1)}{(x-1)(x-5)} = \dfrac{x(x+1)(x-1)}{(x-1)(x-5)} = \dfrac{x(x+1)}{x-5} = g(x)$ for $x \neq 1$.

The graph of g is the same as the graph of f with the exception of a hole in the graph of f at $x = 1$. By long division, $g(x) = \dfrac{x^2 + x}{x - 5} = x + 6 + \dfrac{30}{x-5}$.

As $x \to \pm\infty$, $g(x) \to \pm\infty$, so there is no horizontal asymptote. The denominator of g is zero when $x = 5$. $\lim\limits_{x\to 5^-} g(x) = -\infty$ and $\lim\limits_{x \to 5^+} g(x) = \infty$, so $x = 5$ is a vertical asymptote. The graph confirms our work.

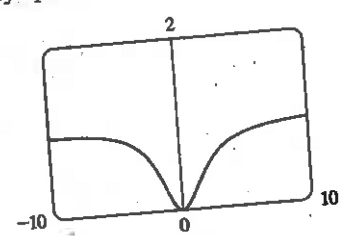


47. From the graph, it appears $y = 1$ is a horizontal asymptote.

$$\lim_{x\to\infty} \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000} = \lim_{x\to\infty} \frac{\dfrac{3x^3 + 500x^2}{x^3}}{\dfrac{x^3 + 500x^2 + 100x + 2000}{x^3}} = \lim_{x\to\infty} \frac{3 + (500/x)}{1 + (500/x) + (100/x^2) + (2000/x^3)}$$

$$= \frac{3+0}{1+0+0+0} = 3, \quad \text{so } y = 3 \text{ is a horizontal asymptote.}$$

The discrepancy can be explained by the choice of the viewing window. Try $[-100{,}000, 100{,}000]$ by $[-1, 4]$ to get a graph that lends credibility to our calculation that $y = 3$ is a horizontal asymptote.



49. Let's look for a rational function.

(1) $\lim\limits_{x\to\pm\infty} f(x) = 0 \;\Rightarrow\;$ degree of numerator < degree of denominator

(2) $\lim\limits_{x\to 0} f(x) = -\infty \;\Rightarrow\;$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim\limits_{x\to 3^-} f(x) = \infty$ and $\lim\limits_{x\to 3^+} f(x) = -\infty \;\Rightarrow\;$ vertical asymptote at $x = 3$; there is a factor of $(x - 3)$ in the denominator.

(4) $f(2) = 0 \;\Rightarrow\;$ 2 is an x-intercept; there is at least one factor of $(x - 2)$ in the numerator.

Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us $f(x) = \dfrac{2 - x}{x^2(x-3)}$ as one possibility.

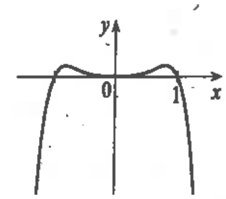
51. (a) We must first find the function f. Since f has a vertical asymptote $x = 4$ and x-intercept $x = 1$, $x - 4$ is a factor of the denominator and $x - 1$ is a factor of the numerator. There is a removable discontinuity at $x = -1$, so $x - (-1) = x + 1$ is a factor of both the numerator and denominator. Thus, f now looks like this: $f(x) = \dfrac{a(x-1)(x+1)}{(x-4)(x+1)}$, where a is still to be determined. Then $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \dfrac{a(x-1)(x+1)}{(x-4)(x+1)} = \lim\limits_{x\to -1} \dfrac{a(x-1)}{x-4} = \dfrac{a(-1-1)}{(-1-4)} = \dfrac{2}{5}a$, so $\dfrac{2}{5}a = 2$, and

$a = 5$. Thus $f(x) = \dfrac{5(x-1)(x+1)}{(x-4)(x+1)}$ is a ratio of quadratic functions satisfying all the given conditions and

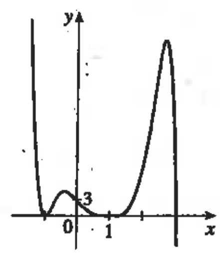
$f(0) = \dfrac{5(-1)(1)}{(-4)(1)} = \dfrac{5}{4}$.

(b) $\lim\limits_{x\to\infty} f(x) = 5 \lim\limits_{x\to\infty} \dfrac{x^2-1}{x^2-3x-4} = 5 \lim\limits_{x\to\infty} \dfrac{(x^2/x^2)-(1/x^2)}{(x^2/x^2)-(3x/x^2)-(4/x^2)} = 5\dfrac{1-0}{1-0-0} = 5(1) = 5$

53. $y = f(x) = x^4 - x^6 = x^4(1-x^2) = x^4(1+x)(1-x)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -1, and 1 [found by solving $f(x) = 0$ for x]. Since $x^4 > 0$ for $x \neq 0$, f doesn't change sign at $x = 0$. The function does change sign at $x = -1$ and $x = 1$. As $x \to \pm\infty$, $f(x) = x^4(1-x^2)$ approaches $-\infty$ because $x^4 \to \infty$ and $(1-x^2) \to -\infty$.

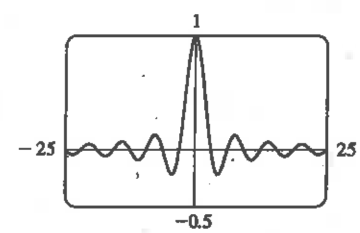


55. $y = f(x) = (3-x)(1+x)^2(1-x)^4$. The y-intercept is $f(0) = 3(1)^2(1)^4 = 3$. The x-intercepts are 3, -1, and 1. There is a sign change at 3, but not at -1 and 1. When x is large positive, $3-x$ is negative and the other factors are positive, so $\lim\limits_{x\to\infty} f(x) = -\infty$. When x is large negative, $3-x$ is positive, so $\lim\limits_{x\to-\infty} f(x) = \infty$.



57. (a) Since $-1 \le \sin x \le 1$ for all x, $-\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$ for $x > 0$. As $x \to \infty$, $-1/x \to 0$ and $1/x \to 0$, so by the Squeeze Theorem, $(\sin x)/x \to 0$. Thus, $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$.

(b) From part (a), the horizontal asymptote is $y = 0$. The function $y = (\sin x)/x$ crosses the horizontal asymptote whenever $\sin x = 0$; that is, at $x = \pi n$ for every integer n. Thus, the graph crosses the asymptote *an infinite number of times.*



59. (a) Divide the numerator and the denominator by the highest power of x in $Q(x)$.

(a) If $\deg P < \deg Q$, then the numerator $\to 0$ but the denominator doesn't. So $\lim\limits_{x\to\infty} [P(x)/Q(x)] = 0$.

(b) If $\deg P > \deg Q$, then the numerator $\to \pm\infty$ but the denominator doesn't, so $\lim\limits_{x\to\infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

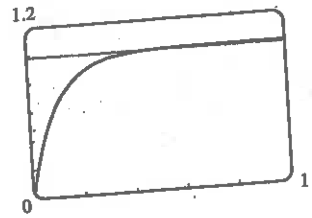
61. $\lim\limits_{x\to\infty} \dfrac{5\sqrt{x}}{\sqrt{x-1}} \cdot \dfrac{1/\sqrt{x}}{1/\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{5}{\sqrt{1-(1/x)}} = \dfrac{5}{\sqrt{1-0}} = 5$ and

$\lim\limits_{x\to\infty} \dfrac{10e^x - 21}{2e^x} \cdot \dfrac{1/e^x}{1/e^x} = \lim\limits_{x\to\infty} \dfrac{10-(21/e^x)}{2} = \dfrac{10-0}{2} = 5$. Since $\dfrac{10e^x-21}{2e^x} < f(x) < \dfrac{5\sqrt{x}}{\sqrt{x-1}}$,

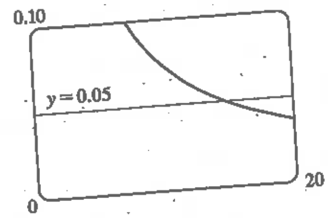
we have $\lim\limits_{x\to\infty} f(x) = 5$ by the Squeeze Theorem.

63. (a) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} v^*\left(1 - e^{-gt/v^*}\right) = v^*(1-0) = v^*$

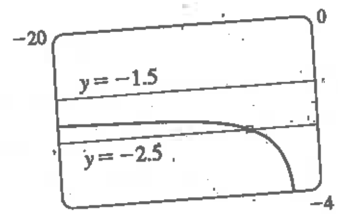
(b) We graph $v(t) = 1 - e^{-9.8t}$ and $v(t) = 0.99v^*$, or in this case, $v(t) = 0.99$. Using an intersect feature or zooming in on the point of intersection, we find that $t \approx 0.47$ s.



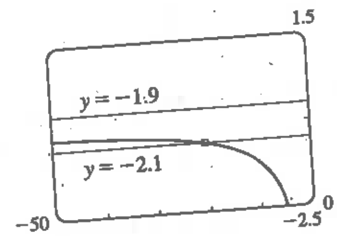
65. Let $g(x) = \dfrac{3x^2+1}{2x^2+x+1}$ and $f(x) = |g(x) - 1.5|$. Note that $\lim_{x\to\infty} g(x) = \frac{3}{2}$ and $\lim_{x\to\infty} f(x) = 0$. We are interested in finding the x-value at which $f(x) < 0.05$. From the graph, we find that $x \approx 14.804$, so we choose $N = 15$ (or any larger number).



67. For $\varepsilon = 0.5$, we need to find N such that $\left|\dfrac{\sqrt{4x^2+1}}{x+1} - (-2)\right| < 0.5 \quad\Leftrightarrow\quad -2.5 < \dfrac{\sqrt{4x^2+1}}{x+1} < -1.5$ whenever $x \le N$. We graph the three parts of this inequality on the same screen, and see that the inequality holds for $x \le -6$. So we choose $N = -6$ (or any smaller number).



For $\varepsilon = 0.1$, we need $-2.1 < \dfrac{\sqrt{4x^2+1}}{x+1} < -1.9$ whenever $x \le N$. From the graph, it seems that this inequality holds for $x \le -22$. So we choose $N = -22$ (or any smaller number).



69. (a) $1/x^2 < 0.0001 \quad\Leftrightarrow\quad x^2 > 1/0.0001 = 10\,000 \quad\Leftrightarrow\quad x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \quad\Leftrightarrow\quad x^2 > 1/\varepsilon \quad\Leftrightarrow\quad x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

Then $x > N \quad\Rightarrow\quad x > \dfrac{1}{\sqrt{\varepsilon}} \quad\Rightarrow\quad \left|\dfrac{1}{x^2} - 0\right| = \dfrac{1}{x^2} < \varepsilon$, so $\lim_{x\to\infty} \dfrac{1}{x^2} = 0$.

71. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \quad\Leftrightarrow\quad x < -1/\varepsilon$.

Take $N = -1/\varepsilon$. Then $x < N \quad\Rightarrow\quad x < -1/\varepsilon \quad\Rightarrow\quad |(1/x) - 0| = -1/x < \varepsilon$, so $\lim_{x\to-\infty} (1/x) = 0$.

73. Given $M > 0$, we need $N > 0$ such that $x > N \quad\Rightarrow\quad e^x > M$. Now $e^x > M \quad\Leftrightarrow\quad x > \ln M$, so take $N = \max(1, \ln M)$. (This ensures that $N > 0$.) Then $x > N = \max(1, \ln M) \quad\Rightarrow\quad e^x > \max(e, M) \ge M$, so $\lim_{x\to\infty} e^x = \infty$.

75. Suppose that $\lim_{x\to\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \quad\Leftrightarrow\quad 0 < 1/x < 1/N \quad\Leftrightarrow\quad 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that $\lim_{t\to 0^+} f(1/t) = L = \lim_{x\to\infty} f(x)$.

Now suppose that $\lim_{x\to-\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \iff 1/N < 1/x < 0 \iff 1/N < t < 0$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\lim_{t\to 0^-} f(1/t) = L = \lim_{x\to-\infty} f(x)$.

2.7 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim_{x\to 3} \dfrac{f(x) - f(3)}{x - 3}$.

3. (a) (i) Using Definition 1 with $f(x) = 4x - x^2$ and $P(1, 3)$,

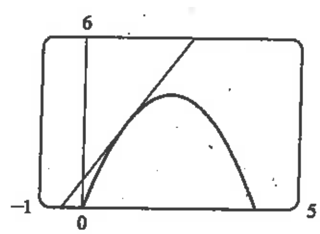
$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 1} \frac{(4x - x^2) - 3}{x - 1} = \lim_{x\to 1} \frac{-(x^2 - 4x + 3)}{x - 1} = \lim_{x\to 1} \frac{-(x - 1)(x - 3)}{x - 1}$$
$$= \lim_{x\to 1}(3 - x) = 3 - 1 = 2$$

(ii) Using Equation 2 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h\to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h\to 0} \frac{[4(1 + h) - (1 + h)^2] - 3}{h}$$
$$= \lim_{h\to 0} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h\to 0} \frac{-h^2 + 2h}{h} = \lim_{h\to 0} \frac{h(-h + 2)}{h} = \lim_{h\to 0}(-h + 2) = 2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \Rightarrow y - f(1) = f'(1)(x - 1) \Rightarrow y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(c)



The graph of $y = 2x + 1$ is tangent to the graph of $y = 4x - x^2$ at the point $(1, 3)$. Now zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistiguishable.

5. Using (1) with $f(x) = 4x - 3x^2$ and $P(2, -4)$ [we could also use (2)],

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 2} \frac{(4x - 3x^2) - (-4)}{x - 2} = \lim_{x\to 2} \frac{-3x^2 + 4x + 4}{x - 2}$$
$$= \lim_{x\to 2} \frac{(-3x - 2)(x - 2)}{x - 2} = \lim_{x\to 2}(-3x - 2) = -3(2) - 2 = -8$$

Tangent line: $y - (-4) = -8(x - 2) \iff y + 4 = -8x + 16 \iff y = -8x + 12$.