

2.5 Continuity

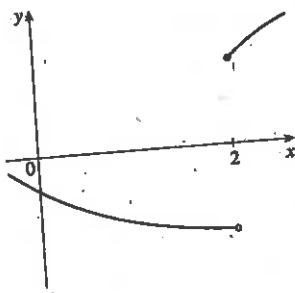
1. From Definition 1, $\lim_{x \rightarrow 4} f(x) = f(4)$.

3. (a) f is discontinuous at -4 since $f(-4)$ is not defined and at $-2, 2,$ and 4 since the limit does not exist (the left and right limits are not the same).

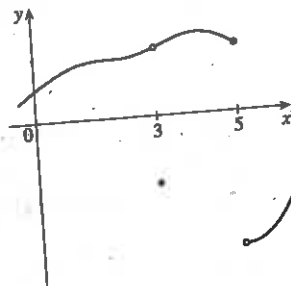
(b) f is continuous from the left at -2 since $\lim_{x \rightarrow -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since

$$\lim_{x \rightarrow 2^+} f(x) = f(2) \text{ and } \lim_{x \rightarrow 4^+} f(x) = f(4).$$

5. The graph of $y = f(x)$ must have a discontinuity at $x = 2$ and must show that $\lim_{x \rightarrow 2^+} f(x) = f(2)$.

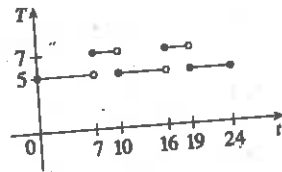


7. The graph of $y = f(x)$ must have a removable discontinuity (a hole) at $x = 3$ and a jump discontinuity at $x = 5$.



9. (a) The toll is \$7 between 7:00 AM and 10:00 AM and between 4:00 PM and 7:00 PM.

(b) The function T has jump discontinuities at $t = 7, 10, 16,$ and 19 . Their significance to someone who uses the road is that, because of the sudden jumps in the toll, they may want to avoid the higher rates between $t = 7$ and $t = 10$ and between $t = 16$ and $t = 19$ if feasible.



11. If f and g are continuous and $g(2) = 6$, then $\lim_{x \rightarrow 2} [3f(x) + f(x)g(x)] = 36 \Rightarrow$

$$3 \lim_{x \rightarrow 2} f(x) + \lim_{x \rightarrow 2} f(x) \cdot \lim_{x \rightarrow 2} g(x) = 36 \Rightarrow 3f(2) + f(2) \cdot 6 = 36 \Rightarrow 9f(2) = 36 \Rightarrow f(2) = 4.$$

$$13. \lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} (x + 2x^3)^4 = \left(\lim_{x \rightarrow -1} x + 2 \lim_{x \rightarrow -1} x^3 \right)^4 = [-1 + 2(-1)^3]^4 = (-3)^4 = 81 = f(-1).$$

By the definition of continuity, f is continuous at $a = -1$.

15. For $a > 2$, we have

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \frac{2x + 3}{x - 2} = \frac{\lim_{x \rightarrow a} (2x + 3)}{\lim_{x \rightarrow a} (x - 2)} \quad \text{[Limit Law 5]}$$

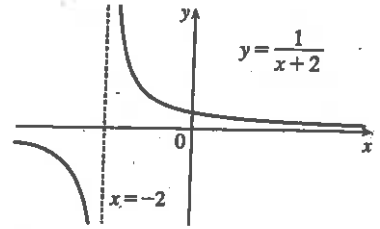
$$= \frac{2 \lim_{x \rightarrow a} x + \lim_{x \rightarrow a} 3}{\lim_{x \rightarrow a} x - \lim_{x \rightarrow a} 2} \quad \text{[1, 2, and 3]}$$

$$= \frac{2a + 3}{a - 2} \quad \text{[7 and 8]}$$

$$= f(a)$$

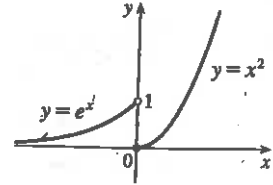
Thus, f is continuous at $x = a$ for every a in $(2, \infty)$; that is, f is continuous on $(2, \infty)$.

17. $f(x) = \frac{1}{x+2}$ is discontinuous at $a = -2$ because $f(-2)$ is undefined.



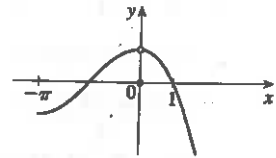
$$19. f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

The left-hand limit of f at $a = 0$ is $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} e^x = 1$. The right-hand limit of f at $a = 0$ is $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 = 0$. Since these limits are not equal, $\lim_{x \rightarrow 0} f(x)$ does not exist and f is discontinuous at 0.



$$21. f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$$

$\lim_{x \rightarrow 0} f(x) = 1$, but $f(0) = 0 \neq 1$, so f is discontinuous at 0.



23. $f(x) = \frac{x^2 - x - 2}{x - 2} = \frac{(x-2)(x+1)}{x-2} = x+1$ for $x \neq 2$. Since $\lim_{x \rightarrow 2} f(x) = 2+1 = 3$, define $f(2) = 3$. Then f is continuous at 2.

25. $F(x) = \frac{x}{x^2 + 5x + 6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\{x \mid x^2 + 5x + 6 \neq 0\} = \{x \mid (x+3)(x+2) \neq 0\} = \{x \mid x \neq -3, -2\}$ or $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

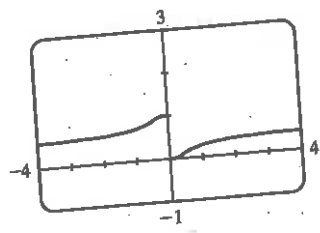
27. By Theorem 5, the polynomials x^2 and $2x - 1$ are continuous on $(-\infty, \infty)$. By Theorem 7, the root function \sqrt{x} is continuous on $[0, \infty)$. By Theorem 9, the composite function $\sqrt{2x-1}$ is continuous on its domain, $[\frac{1}{2}, \infty)$.

By part 1 of Theorem 4, the sum $R(x) = x^2 + \sqrt{2x-1}$ is continuous on $[\frac{1}{2}, \infty)$.

29. By Theorem 5(a), the polynomial $1 + 2t$ is continuous on \mathbb{R} . By Theorem 7, the inverse trigonometric function $\arcsin x$ is continuous on its domain, $[-1, 1]$. By Theorem 9, $A(t) = \arcsin(1 + 2t)$ is continuous on its domain, which is $\{t \mid -1 \leq 1 + 2t \leq 1\} = \{t \mid -2 \leq 2t \leq 0\} = \{t \mid -1 \leq t \leq 0\} = [-1, 0]$.

31. $M(x) = \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}}$ is defined when $\frac{x+1}{x} \geq 0 \Rightarrow x+1 \geq 0$ and $x > 0$ or $x+1 \leq 0$ and $x < 0 \Rightarrow x > 0$ or $x \leq -1$, so M has domain $(-\infty, -1] \cup (0, \infty)$. M is the composite of a root function and a rational function, so it is continuous at every number in its domain by Theorems 7 and 9.

33. The function $y = \frac{1}{1 + e^{1/x}}$ is discontinuous at $x = 0$ because the left- and right-hand limits at $x = 0$ are different.



35. Because we are dealing with root functions, $5 + \sqrt{x}$ is continuous on $[0, \infty)$, $\sqrt{x+5}$ is continuous on $[-5, \infty)$, so the quotient $f(x) = \frac{5 + \sqrt{x}}{\sqrt{5+x}}$ is continuous on $[0, \infty)$. Since f is continuous at $x = 4$, $\lim_{x \rightarrow 4} f(x) = f(4) = \frac{7}{3}$.

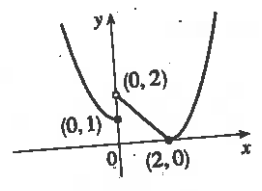
37. Because $x^2 - x$ is continuous on \mathbb{R} , the composite function $f(x) = e^{x^2 - x}$ is continuous on \mathbb{R} , so $\lim_{x \rightarrow 1} f(x) = f(1) = e^{1-1} = e^0 = 1$.

$$39. f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty, 1)$, f is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function \sqrt{x} on $(1, \infty)$, f is continuous on $(1, \infty)$. At $x = 1$, $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \sqrt{x} = 1$. Thus, $\lim_{x \rightarrow 1} f(x)$ exists and equals 1. Also, $f(1) = \sqrt{1} = 1$. Thus, f is continuous at $x = 1$.

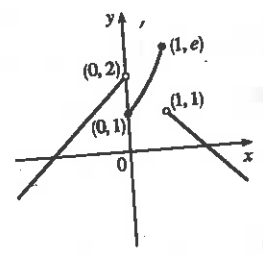
We conclude that f is continuous on $(-\infty, \infty)$.

$$41. f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$$



f is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on each of these intervals. Now $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (1 + x^2) = 1$ and $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (2 - x) = 2$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the left at 0. Also, $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2 - x) = 0$, $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x-2)^2 = 0$, and $f(2) = 0$, so f is continuous at 2. The only number at which f is discontinuous is 0.

$$43. f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$



f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0, 1)$ since it is an exponential. Now $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x + 2) = 2$ and $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^x = 1$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0. Also $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} e^x = e$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2 - x) = 1$, so f is discontinuous at 1. Since $f(1) = e$, f is continuous from the left at 1.

$$45. f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

f is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (cx^2 + 2x) = 4c + 4$ and

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^3 - cx) = 8 - 2c. \text{ So } f \text{ is continuous} \Leftrightarrow 4c + 4 = 8 - 2c \Leftrightarrow 6c = 4 \Leftrightarrow c = \frac{2}{3}. \text{ Thus, for } f$$

to be continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$.

$$47. (a) f(x) = \frac{x^4 - 1}{x - 1} = \frac{(x^2 + 1)(x^2 - 1)}{x - 1} = \frac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = (x^2 + 1)(x + 1) \quad [\text{or } x^3 + x^2 + x + 1]$$

for $x \neq 1$. The discontinuity is removable and $g(x) = x^3 + x^2 + x + 1$ agrees with f for $x \neq 1$ and is continuous on \mathbb{R} .

$$(b) f(x) = \frac{x^3 - x^2 - 2x}{x - 2} = \frac{x(x^2 - x - 2)}{x - 2} = \frac{x(x - 2)(x + 1)}{x - 2} = x(x + 1) \quad [\text{or } x^2 + x] \text{ for } x \neq 2. \text{ The discontinuity}$$

is removable and $g(x) = x^2 + x$ agrees with f for $x \neq 2$ and is continuous on \mathbb{R} .

$$(c) \lim_{x \rightarrow \pi^-} f(x) = \lim_{x \rightarrow \pi^-} [\sin x] = \lim_{x \rightarrow \pi^-} 0 = 0 \text{ and } \lim_{x \rightarrow \pi^+} f(x) = \lim_{x \rightarrow \pi^+} [\sin x] = \lim_{x \rightarrow \pi^+} (-1) = -1, \text{ so } \lim_{x \rightarrow \pi} f(x) \text{ does not exist. The discontinuity at } x = \pi \text{ is a jump discontinuity.}$$

49. $f(x) = x^2 + 10 \sin x$ is continuous on the interval $[31, 32]$, $f(31) \approx 957$, and $f(32) \approx 1030$. Since $957 < 1000 < 1030$, there is a number c in $(31, 32)$ such that $f(c) = 1000$ by the Intermediate Value Theorem. Note: There is also a number c in $(-32, -31)$ such that $f(c) = 1000$.

51. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

53. The equation $e^x = 3 - 2x$ is equivalent to the equation $e^x + 2x - 3 = 0$. $f(x) = e^x + 2x - 3$ is continuous on the interval $[0, 1]$, $f(0) = -2$, and $f(1) = e - 1 \approx 1.72$. Since $-2 < 0 < e - 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^x + 2x - 3 = 0$, or $e^x = 3 - 2x$, in the interval $(0, 1)$.

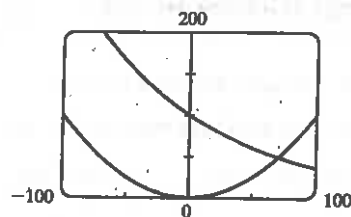
55. (a) $f(x) = \cos x - x^3$ is continuous on the interval $[0, 1]$, $f(0) = 1 > 0$, and $f(1) = \cos 1 - 1 \approx -0.46 < 0$. Since $1 > 0 > -0.46$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x^3 = 0$, or $\cos x = x^3$, in the interval $(0, 1)$.

(b) $f(0.86) \approx 0.016 > 0$ and $f(0.87) \approx -0.014 < 0$, so there is a root between 0.86 and 0.87, that is, in the interval $(0.86, 0.87)$.

57. (a) Let $f(x) = 100e^{-x/100} - 0.01x^2$. Then $f(0) = 100 > 0$ and $f(100) = 100e^{-1} - 100 \approx -63.2 < 0$. So by the Intermediate Value Theorem, there is a number c in $(0, 100)$ such that $f(c) = 0$.

This implies that $100e^{-c/100} = 0.01c^2$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 70.347$, correct to three decimal places.



59. (\Rightarrow) If f is continuous at a , then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h \rightarrow 0} f(a+h) = f\left(\lim_{h \rightarrow 0} (a+h)\right) = f(a).$$

(\Leftarrow) Let $\varepsilon > 0$. Since $\lim_{h \rightarrow 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \Rightarrow$

$$|f(a+h) - f(a)| < \varepsilon. \text{ So if } 0 < |x-a| < \delta, \text{ then } |f(x) - f(a)| = |f(a+(x-a)) - f(a)| < \varepsilon.$$

Thus, $\lim_{x \rightarrow a} f(x) = f(a)$ and so f is continuous at a .

61. As in the previous exercise, we must show that $\lim_{h \rightarrow 0} \cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\begin{aligned} \lim_{h \rightarrow 0} \cos(a+h) &= \lim_{h \rightarrow 0} (\cos a \cos h - \sin a \sin h) = \lim_{h \rightarrow 0} (\cos a \cos h) - \lim_{h \rightarrow 0} (\sin a \sin h) \\ &= \left(\lim_{h \rightarrow 0} \cos a\right) \left(\lim_{h \rightarrow 0} \cos h\right) - \left(\lim_{h \rightarrow 0} \sin a\right) \left(\lim_{h \rightarrow 0} \sin h\right) = (\cos a)(1) - (\sin a)(0) = \cos a \end{aligned}$$

63. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a - \delta, a + \delta)$

contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1 , there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x \rightarrow a} f(x) \neq f(a)$. [In fact, $\lim_{x \rightarrow a} f(x)$ does not even exist.]

65. If there is such a number, it satisfies the equation $x^3 + 1 = x \Leftrightarrow x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

67. $f(x) = x^4 \sin(1/x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$ since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since $-1 \leq \sin(1/x) \leq 1$, we have $-x^4 \leq x^4 \sin(1/x) \leq x^4$. Because $\lim_{x \rightarrow 0} (-x^4) = 0$ and $\lim_{x \rightarrow 0} x^4 = 0$, the Squeeze Theorem gives us $\lim_{x \rightarrow 0} (x^4 \sin(1/x)) = 0$, which equals $f(0)$. Thus, f is continuous at 0 and, hence, on $(-\infty, \infty)$.

69. Define $u(t)$ to be the monk's distance from the monastery, as a function of time t (in hours), on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \Leftrightarrow u(t_0) = d(t_0)$. So at time t_0 after 7:00 AM, the monk will be at the same place on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

1. (a) As x becomes large, the values of $f(x)$ approach 5.

(b) As x becomes large negative, the values of $f(x)$ approach 3.

3. (a) $\lim_{x \rightarrow 2} f(x) = \infty$

(b) $\lim_{x \rightarrow -1^-} f(x) = \infty$

(c) $\lim_{x \rightarrow -1^+} f(x) = -\infty$

(d) $\lim_{x \rightarrow \infty} f(x) = 1$

(e) $\lim_{x \rightarrow -\infty} f(x) = 2$

(f) Vertical: $x = -1, x = 2$; Horizontal: $y = 1, y = 2$