

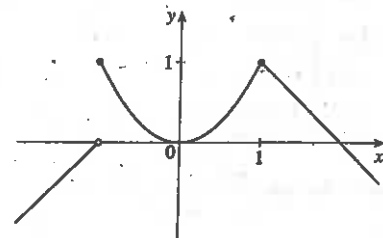
2.2 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.
3. (a) $\lim_{x \rightarrow -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).
- (b) $\lim_{x \rightarrow 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.
5. (a) As x approaches 1, the values of $f(x)$ approach 2, so $\lim_{x \rightarrow 1} f(x) = 2$.
- (b) As x approaches 3 from the left, the values of $f(x)$ approach 1, so $\lim_{x \rightarrow 3^-} f(x) = 1$.
- (c) As x approaches 3 from the right, the values of $f(x)$ approach 4, so $\lim_{x \rightarrow 3^+} f(x) = 4$.
- (d) $\lim_{x \rightarrow 3} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.
- (e) When $x = 3$, $y = 3$, so $f(3) = 3$.
7. (a) $\lim_{t \rightarrow 0^-} g(t) = -1$ (b) $\lim_{t \rightarrow 0^+} g(t) = -2$
- (c) $\lim_{t \rightarrow 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.
- (d) $\lim_{t \rightarrow 2^-} g(t) = 2$ (e) $\lim_{t \rightarrow 2^+} g(t) = 0$
- (f) $\lim_{t \rightarrow 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.
- (g) $g(2) = 1$ (h) $\lim_{t \rightarrow 4} g(t) = 3$
9. (a) $\lim_{x \rightarrow -7} f(x) = -\infty$ (b) $\lim_{x \rightarrow -3} f(x) = \infty$ (c) $\lim_{x \rightarrow 0} f(x) = \infty$
- (d) $\lim_{x \rightarrow 6^-} f(x) = -\infty$ (e) $\lim_{x \rightarrow 6^+} f(x) = \infty$
- (f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

11. From the graph of

$$f(x) = \begin{cases} 1+x & \text{if } x < -1, \\ x^2 & \text{if } -1 \leq x < 1, \\ 2-x & \text{if } x \geq 1 \end{cases}$$

we see that $\lim_{x \rightarrow a} f(x)$ exists for all a except $a = -1$. Notice that the right and left limits are different at $a = -1$.

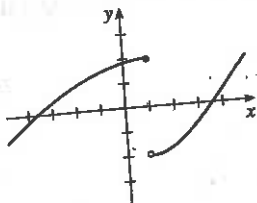


13. (a) $\lim_{x \rightarrow 0^-} f(x) = 1$

(b) $\lim_{x \rightarrow 0^+} f(x) = 0$

(c) $\lim_{x \rightarrow 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

15. $\lim_{x \rightarrow 1^-} f(x) = 2$, $\lim_{x \rightarrow 1^+} f(x) = -2$, $f(1) = 2$



19. For $f(x) = \frac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
2.5	0.714286
2.1	0.677419
2.05	0.672131
2.01	0.667774
2.005	0.667221
2.001	0.666778

x	$f(x)$
1.9	0.655172
1.95	0.661017
1.99	0.665552
1.995	0.666110
1.999	0.666556

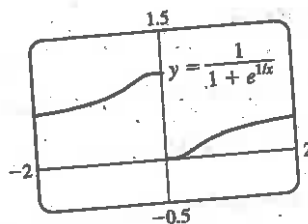
It appears that $\lim_{x \rightarrow 2} \frac{x^2 - 2x}{x^2 - x - 2} = 0.6 = \frac{2}{3}$.

23. For $f(x) = \frac{\sqrt{x+4} - 2}{x}$:

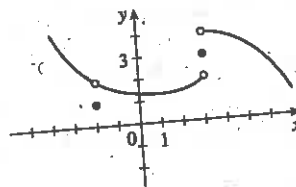
x	$f(x)$
1	0.236068
0.5	0.242641
0.1	0.248457
0.05	0.249224
0.01	0.249844

x	$f(x)$
-1	0.267949
-0.5	0.258343
-0.1	0.251582
-0.05	0.250786
-0.01	0.250156

It appears that $\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x} = 0.25 = \frac{1}{4}$.



17. $\lim_{x \rightarrow 3^+} f(x) = 4$, $\lim_{x \rightarrow 3^-} f(x) = 2$, $\lim_{x \rightarrow -2} f(x) = 2$, $f(3) = 3$, $f(-2) = 1$



21. For $f(x) = \frac{e^x - 1 - x}{x^2}$:

x	$f(x)$
1	0.718282
0.5	0.594885
0.1	0.517092
0.05	0.508439
0.01	0.501671

x	$f(x)$
-1	0.367879
-0.5	0.426123
-0.1	0.483742
-0.05	0.491770
-0.01	0.498337

It appears that $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = 0.5 = \frac{1}{2}$.

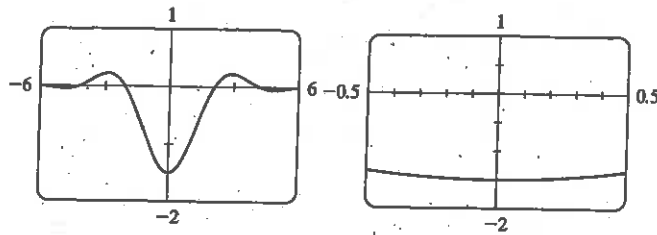
25. For $f(x) = \frac{x^6 - 1}{x^{10} - 1}$:

x	$f(x)$
0.5	0.985337
0.9	0.719397
0.95	0.660186
0.99	0.612018
0.999	0.601200

x	$f(x)$
1.5	0.183369
1.1	0.484119
1.05	0.540783
1.01	0.588022
1.001	0.598800

It appears that $\lim_{x \rightarrow 1} \frac{x^6 - 1}{x^{10} - 1} = 0.6 = \frac{3}{5}$.

27. (a) From the graphs, it seems that $\lim_{x \rightarrow 0} \frac{\cos 2x - \cos x}{x^2} = -1.5$.



(b)

x	$f(x)$
± 0.1	-1.493759
± 0.01	-1.499938
± 0.001	-1.499999
± 0.0001	-1.500000

29. $\lim_{x \rightarrow -3^+} \frac{x+2}{x+3} = -\infty$ since the numerator is negative and the denominator approaches 0 from the positive side as $x \rightarrow -3^+$.

31. $\lim_{x \rightarrow 1} \frac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \rightarrow 1$.

33. Let $t = x^2 - 9$. Then as $x \rightarrow 3^+$, $t \rightarrow 0^+$, and $\lim_{x \rightarrow 3^+} \ln(x^2 - 9) = \lim_{t \rightarrow 0^+} \ln t = -\infty$ by (5).

35. $\lim_{x \rightarrow 2\pi^-} x \csc x = \lim_{x \rightarrow 2\pi^-} \frac{x}{\sin x} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \rightarrow 2\pi^-$.

37. $\lim_{x \rightarrow 2^+} \frac{x^2 - 2x - 8}{x^2 - 5x + 6} = \lim_{x \rightarrow 2^+} \frac{(x-4)(x+2)}{(x-3)(x-2)} = \infty$ since the numerator is negative and the denominator approaches 0 through negative values as $x \rightarrow 2^+$.

39. (a) $f(x) = \frac{1}{x^3 - 1}$.

From these calculations, it seems that

$$\lim_{x \rightarrow 1^-} f(x) = -\infty \text{ and } \lim_{x \rightarrow 1^+} f(x) = \infty.$$

x	$f(x)$
0.5	-1.14
0.9	-3.69
0.99	-33.7
0.999	-333.7
0.9999	-3333.7
0.99999	-33,333.7

x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

- (b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim_{x \rightarrow 1^-} f(x) = -\infty$.

If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim_{x \rightarrow 1^+} f(x) = \infty$.

- (c) It appears from the graph of f that

$$\lim_{x \rightarrow 1^-} f(x) = -\infty \text{ and } \lim_{x \rightarrow 1^+} f(x) = \infty.$$

