3.6 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

2–22 Differentiate the function.

2. $f(x) = x \ln x - x$
3. $f(x) = \sin(\ln x)$
4. $f(x) = \ln(\sin^2 x)$
5. $f(x) = \sqrt[5]{\ln x}$
6. $f(x) = \ln \sqrt[5]{x}$
7. $f(x) = \log_{10}(x^3 + 1)$
8. $f(x) = \log_5(xe^x)$
9. $f(x) = \sin x \ln(5x)$
10. $f(u) = \dfrac{u}{1 + \ln u}$
11. $g(x) = \ln\left(x\sqrt{x^2 - 1}\right)$
12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$
13. $G(y) = \ln \dfrac{(2y + 1)^5}{\sqrt{y^2 + 1}}$
14. $g(r) = r^2 \ln(2r + 1)$
15. $F(s) = \ln \ln s$
16. $y = \ln|1 + t - t^3|$
17. $y = \tan[\ln(ax + b)]$
18. $y = \ln|\cos(\ln x)|$
19. $y = \ln(e^{-x} + xe^{-x})$
20. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$
21. $y = 2x \log_{10}\sqrt{x}$
22. $y = \log_2(e^{-x} \cos \pi x)$

23–26 Find y' and y''.

23. $y = x^2 \ln(2x)$
24. $y = \dfrac{\ln x}{x^2}$
25. $y = \ln\left(x + \sqrt{1 + x^2}\right)$
26. $y = \ln(\sec x + \tan x)$

27–30 Differentiate f and find the domain of f.

27. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$
28. $f(x) = \sqrt{2 + \ln x}$
29. $f(x) = \ln(x^2 - 2x)$
30. $f(x) = \ln \ln \ln x$

31. If $f(x) = \dfrac{\ln x}{x^2}$, find $f'(1)$.

32. If $f(x) = \ln(1 + e^{2x})$, find $f'(0)$.

33–34 Find an equation of the tangent line to the curve at the given point.

33. $y = \ln(x^2 - 3x + 1)$, $(3, 0)$
34. $y = x^2 \ln x$, $(1, 0)$

35. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

36. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

37. Let $f(x) = cx + \ln(\cos x)$. For what value of c is $f'(\pi/4) = 6$?

38. Let $f(x) = \log_a(3x^2 - 2)$. For what value of a is $f'(1) = 3$?

39–50 Use logarithmic differentiation to find the derivative of the function.

39. $y = (2x + 1)^5(x^4 - 3)^6$
40. $y = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}$
41. $y = \sqrt{\dfrac{x - 1}{x^4 + 1}}$
42. $y = \sqrt{x}\, e^{x^2 - x}(x + 1)^{2/3}$
43. $y = x^x$
44. $y = x^{\cos x}$
45. $y = x^{\sin x}$
46. $y = \sqrt{x}^{\,x}$
47. $y = (\cos x)^x$
48. $y = (\sin x)^{\ln x}$
49. $y = (\tan x)^{1/x}$
50. $y = (\ln x)^{\cos x}$

51. Find y' if $y = \ln(x^2 + y^2)$.

52. Find y' if $x^y = y^x$.

53. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

54. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

55. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

56. Show that $\displaystyle\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

Graphing calculator or computer required

1. Homework Hints available at stewartcalculus.com

1. The differentiation formula for logarithmic functions, $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$, is simplest when $a = e$ because $\ln e = 1$.

3. $f(x) = \sin(\ln x) \quad \Rightarrow \quad f'(x) = \cos(\ln x) \cdot \frac{d}{dx} \ln x = \cos(\ln x) \cdot \frac{1}{x} = \frac{\cos(\ln x)}{x}$

5. $f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \quad \Rightarrow \quad f'(x) = \frac{1}{5}(\ln x)^{-4/5} \frac{d}{dx}(\ln x) = \frac{1}{5(\ln x)^{4/5}} \cdot \frac{1}{x} = \frac{1}{5x \sqrt[5]{(\ln x)^4}}$

7. $f(x) = \log_{10}(x^3 + 1) \quad \Rightarrow \quad f'(x) = \frac{1}{(x^3+1)\ln 10} \frac{d}{dx}(x^3 + 1) = \frac{3x^2}{(x^3+1)\ln 10}$

9. $f(x) = \sin x \ln(5x) \quad \Rightarrow \quad f'(x) = \sin x \cdot \frac{1}{5x} \cdot \frac{d}{dx}(5x) + \ln(5x) \cdot \cos x = \frac{\sin x \cdot 5}{5x} + \cos x \ln(5x) = \frac{\sin x}{x} + \cos x \ln(5x)$

11. $g(x) = \ln\left(x\sqrt{x^2 - 1}\right) = \ln x + \ln(x^2 - 1)^{1/2} = \ln x + \frac{1}{2}\ln(x^2 - 1) \quad \Rightarrow$

$g'(x) = \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x^2 - 1} \cdot 2x = \frac{1}{x} + \frac{x}{x^2 - 1} = \frac{x^2 - 1 + x \cdot x}{x(x^2 - 1)} = \frac{2x^2 - 1}{x(x^2 - 1)}$

13. $G(y) = \ln \frac{(2y+1)^5}{\sqrt{y^2 + 1}} = \ln(2y+1)^5 - \ln(y^2+1)^{1/2} = 5\ln(2y+1) - \frac{1}{2}\ln(y^2+1) \quad \Rightarrow$

$G'(y) = 5 \cdot \frac{1}{2y+1} \cdot 2 - \frac{1}{2} \cdot \frac{1}{y^2+1} \cdot 2y = \frac{10}{2y+1} - \frac{y}{y^2+1}$ $\left[\text{or } \frac{8y^2 - y + 10}{(2y+1)(y^2+1)}\right]$

15. $F(s) = \ln \ln s \quad \Rightarrow \quad F'(s) = \frac{1}{\ln s} \frac{d}{ds} \ln s = \frac{1}{\ln s} \cdot \frac{1}{s} = \frac{1}{s \ln s}$

17. $y = \tan[\ln(ax + b)] \quad \Rightarrow \quad y' = \sec^2[\ln(ax+b)] \cdot \frac{1}{ax+b} \cdot a = \sec^2[\ln(ax+b)] \frac{a}{ax+b}$

19. $y = \ln(e^{-x} + xe^{-x}) = \ln(e^{-x}(1+x)) = \ln(e^{-x}) + \ln(1+x) = -x + \ln(1+x) \quad \Rightarrow$

$y' = -1 + \frac{1}{1+x} = \frac{-1 - x + 1}{1+x} = -\frac{x}{1+x}$

21. $y = 2x \log_{10} \sqrt{x} = 2x \log_{10} x^{1/2} = 2x \cdot \frac{1}{2} \log_{10} x = x \log_{10} x \quad \Rightarrow \quad y' = x \cdot \frac{1}{x \ln 10} + \log_{10} x \cdot 1 = \frac{1}{\ln 10} + \log_{10} x$

Note: $\frac{1}{\ln 10} = \frac{\ln e}{\ln 10} = \log_{10} e$, so the answer could be written as $\frac{1}{\ln 10} + \log_{10} x = \log_{10} e + \log_{10} x = \log_{10} ex$.

23. $y = x^2 \ln(2x) \quad \Rightarrow \quad y' = x^2 \cdot \frac{1}{2x} \cdot 2 + \ln(2x) \cdot (2x) = x + 2x\ln(2x) \quad \Rightarrow$

$y'' = 1 + 2x \cdot \frac{1}{2x} \cdot 2 + \ln(2x) \cdot 2 = 1 + 2 + 2\ln(2x) = 3 + 2\ln(2x)$

25. $y = \ln\left(x + \sqrt{1+x^2}\right) \Rightarrow$

$$y' = \frac{1}{x+\sqrt{1+x^2}}\frac{d}{dx}\left(x+\sqrt{1+x^2}\right) = \frac{1}{x+\sqrt{1+x^2}}\left[1+\tfrac{1}{2}(1+x^2)^{-1/2}(2x)\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right) = \frac{1}{x+\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} \Rightarrow$$

$$y'' = -\tfrac{1}{2}(1+x^2)^{-3/2}(2x) = \frac{-x}{(1+x^2)^{3/2}}$$

27. $f(x) = \dfrac{x}{1-\ln(x-1)} \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot\dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$

$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$$\text{Dom}(f) = \{x \mid x-1>0 \text{ and } 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \text{ and } \ln(x-1)\neq 1\}$$

$$= \{x \mid x>1 \text{ and } x-1\neq e^1\} = \{x \mid x>1 \text{ and } x\neq 1+e\} = (1,1+e)\cup(1+e,\infty)$$

29. $f(x) = \ln(x^2-2x) \Rightarrow f'(x) = \dfrac{1}{x^2-2x}(2x-2) = \dfrac{2(x-1)}{x(x-2)}$.

$\text{Dom}(f) = \{x \mid x(x-2)>0\} = (-\infty,0)\cup(2,\infty)$.

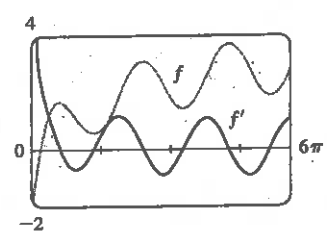
31. $f(x) = \dfrac{\ln x}{x^2} \Rightarrow f'(x) = \dfrac{x^2(1/x)-(\ln x)(2x)}{(x^2)^2} = \dfrac{x-2x\ln x}{x^4} = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3}$,

so $f'(1) = \dfrac{1-2\ln 1}{1^3} = \dfrac{1-2\cdot 0}{1} = 1$.

33. $y = \ln(x^2-3x+1) \Rightarrow y' = \dfrac{1}{x^2-3x+1}\cdot(2x-3) \Rightarrow y'(3) = \frac{1}{1}\cdot 3 = 3$, so an equation of a tangent line at $(3,0)$ is $y-0 = 3(x-3)$, or $y = 3x-9$.

35. $f(x) = \sin x + \ln x \Rightarrow f'(x) = \cos x + 1/x$.

This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



37. $f(x) = cx + \ln(\cos x) \Rightarrow f'(x) = c + \dfrac{1}{\cos x}\cdot(-\sin x) = c - \tan x$.

$f'(\frac{\pi}{4}) = 6 \Rightarrow c - \tan\frac{\pi}{4} = 6 \Rightarrow c-1=6 \Rightarrow c=7$.

39. $y = (2x+1)^5(x^4-3)^6 \Rightarrow \ln y = \ln\left((2x+1)^5(x^4-3)^6\right) \Rightarrow \ln y = 5\ln(2x+1) + 6\ln(x^4-3) \Rightarrow$

$\frac{1}{y}y' = 5\cdot\frac{1}{2x+1}\cdot 2 + 6\cdot\frac{1}{x^4-3}\cdot 4x^3 \Rightarrow$

$y' = y\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right) = (2x+1)^5(x^4-3)^6\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right).$

[The answer could be simplified to $y' = 2(2x+1)^4(x^4-3)^5(29x^4+12x^3-15)$, but this is unnecessary.]

41. $y = \sqrt{\frac{x-1}{x^4+1}} \Rightarrow \ln y = \ln\left(\frac{x-1}{x^4+1}\right)^{1/2} \Rightarrow \ln y = \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x^4+1) \Rightarrow$

$\frac{1}{y}y' = \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x^4+1}\cdot 4x^3 \Rightarrow y' = y\left(\frac{1}{2(x-1)} - \frac{2x^3}{x^4+1}\right) \Rightarrow y' = \sqrt{\frac{x-1}{x^4+1}}\left(\frac{1}{2x-2} - \frac{2x^3}{x^4+1}\right)$

43. $y = x^x \Rightarrow \ln y = \ln x^x \Rightarrow \ln y = x\ln x \Rightarrow y'/y = x(1/x) + (\ln x)\cdot 1 \Rightarrow y' = y(1+\ln x) \Rightarrow$

$y' = x^x(1+\ln x)$

45. $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} \Rightarrow \ln y = \sin x \ln x \Rightarrow \frac{y'}{y} = (\sin x)\cdot\frac{1}{x} + (\ln x)(\cos x) \Rightarrow$

$y' = y\left(\frac{\sin x}{x} + \ln x\cos x\right) \Rightarrow y' = x^{\sin x}\left(\frac{\sin x}{x} + \ln x\cos x\right)$

47. $y = (\cos x)^x \Rightarrow \ln y = \ln(\cos x)^x \Rightarrow \ln y = x\ln\cos x \Rightarrow \frac{1}{y}y' = x\cdot\frac{1}{\cos x}\cdot(-\sin x) + \ln\cos x\cdot 1 \Rightarrow$

$y' = y\left(\ln\cos x - \frac{x\sin x}{\cos x}\right) \Rightarrow y' = (\cos x)^x(\ln\cos x - x\tan x)$

49. $y = (\tan x)^{1/x} \Rightarrow \ln y = \ln(\tan x)^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln\tan x \Rightarrow$

$\frac{1}{y}y' = \frac{1}{x}\cdot\frac{1}{\tan x}\cdot\sec^2 x + \ln\tan x\cdot\left(-\frac{1}{x^2}\right) \Rightarrow y' = y\left(\frac{\sec^2 x}{x\tan x} - \frac{\ln\tan x}{x^2}\right) \Rightarrow$

$y' = (\tan x)^{1/x}\left(\frac{\sec^2 x}{x\tan x} - \frac{\ln\tan x}{x^2}\right)$ or $y' = (\tan x)^{1/x}\cdot\frac{1}{x}\left(\csc x\sec x - \frac{\ln\tan x}{x}\right)$

51. $y = \ln(x^2+y^2) \Rightarrow y' = \frac{1}{x^2+y^2}\frac{d}{dx}(x^2+y^2) \Rightarrow y' = \frac{2x+2yy'}{x^2+y^2} \Rightarrow x^2y' + y^2y' = 2x + 2yy' \Rightarrow$

$x^2y' + y^2y' - 2yy' = 2x \Rightarrow (x^2+y^2-2y)y' = 2x \Rightarrow y' = \frac{2x}{x^2+y^2-2y}$

53. $f(x) = \ln(x-1) \Rightarrow f'(x) = \frac{1}{(x-1)} = (x-1)^{-1} \Rightarrow f''(x) = -(x-1)^{-2} \Rightarrow f'''(x) = 2(x-1)^{-3} \Rightarrow$

$f^{(4)}(x) = -2\cdot 3(x-1)^{-4} \Rightarrow \cdots \Rightarrow f^{(n)}(x) = (-1)^{n-1}\cdot 2\cdot 3\cdot 4\cdot\cdots\cdot(n-1)(x-1)^{-n} = (-1)^{n-1}\frac{(n-1)!}{(x-1)^n}$

55. If $f(x) = \ln(1+x)$, then $f'(x) = \frac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = f'(0) = 1.$