so $$\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

As $\Delta x \to 0$, Equation 8 shows that $\Delta u \to 0$. So both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$. Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$
$$= f'(b)g'(a) = f'(g(a))g'(a)$$

This proves the Chain Rule.

3.4 Exercises

1–6 Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx.

1. $y = \sin 4x$
2. $y = \sqrt{4 + 3x}$
3. $y = (1 - x^2)^{10}$
4. $y = \tan(\sin x)$
5. $y = e^{\sqrt{x}}$
6. $y = \sqrt{2 - e^x}$

7–46 Find the derivative of the function.

7. $F(x) = (x^4 + 3x^2 - 2)^5$
8. $F(x) = (4x - x^2)^{100}$
9. $F(x) = \sqrt[4]{1 + 2x + x^3}$
10. $f(x) = (1 + x^4)^{2/3}$
11. $g(t) = \dfrac{1}{(t^4 + 1)^3}$
12. $f(t) = \sqrt[3]{1 + \tan t}$
13. $y = \cos(a^3 + x^3)$
14. $y = a^3 + \cos^3 x$
15. $y = xe^{-kx}$
16. $y = e^{-2t}\cos 4t$
17. $f(x) = (2x - 3)^4(x^2 + x + 1)^5$
18. $g(x) = (x^2 + 1)^3(x^2 + 2)^6$
19. $h(t) = (t + 1)^{2/3}(2t^2 - 1)^3$
20. $F(t) = (3t - 1)^4(2t + 1)^{-3}$
21. $y = \left(\dfrac{x^2 + 1}{x^2 - 1}\right)^3$
22. $f(s) = \sqrt{\dfrac{s^2 + 1}{s^2 + 4}}$
23. $y = \sqrt{1 + 2e^{3x}}$
24. $y = 10^{1-x^2}$
25. $y = 5^{-1/x}$
26. $G(y) = \dfrac{(y - 1)^4}{(y^2 + 2y)^5}$
27. $y = \dfrac{r}{\sqrt{r^2 + 1}}$
28. $y = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$
29. $F(t) = e^{t \sin 2t}$
30. $F(v) = \left(\dfrac{v}{v^3 + 1}\right)^6$
31. $y = \sin(\tan 2x)$
32. $y = \sec^2(m\theta)$
33. $y = 2^{\sin \pi x}$
34. $y = x^2 e^{-1/x}$
35. $y = \cos\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$
36. $y = \sqrt{1 + xe^{-2x}}$
37. $y = \cot^2(\sin\theta)$
38. $y = e^{k \tan \sqrt{x}}$
39. $f(t) = \tan(e^t) + e^{\tan t}$
40. $y = \sin(\sin(\sin x))$
41. $f(t) = \sin^2(e^{\sin^2 t})$
42. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
43. $g(x) = (2ra^{rx} + n)^p$
44. $y = 2^{3^{x^2}}$
45. $y = \cos\sqrt{\sin(\tan \pi x)}$
46. $y = [x + (x + \sin^2 x)^3]^4$

47–50 Find y' and y''.

47. $y = \cos(x^2)$
48. $y = \cos^2 x$
49. $y = e^{\alpha x}\sin \beta x$
50. $y = e^{e^x}$

51–54 Find an equation of the tangent line to the curve at the given point.

51. $y = (1 + 2x)^{10}$, $(0, 1)$
52. $y = \sqrt{1 + x^3}$, $(2, 3)$
53. $y = \sin(\sin x)$, $(\pi, 0)$
54. $y = \sin x + \sin^2 x$, $(0, 0)$

55. (a) Find an equation of the tangent line to the curve $y = 2/(1 + e^{-x})$ at the point $(0, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

56. (a) The curve $y = |x|/\sqrt{2 - x^2}$ is called a *bullet-nose curve*. Find an equation of the tangent line to this curve at the point $(1, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

57. (a) If $f(x) = x\sqrt{2 - x^2}$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

Graphing calculator or computer required   CAS Computer algebra system required   1. Homework Hints available at stewartcalculus.com

58. The function $f(x) = \sin(x + \sin 2x)$, $0 \leq x \leq \pi$, arises in applications to frequency modulation (FM) synthesis.
(a) Use a graph of f produced by a graphing device to make a rough sketch of the graph of f'.
(b) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (a).

59. Find all points on the graph of the function $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.

60. Find the x-coordinates of all points on the curve $y = \sin 2x - 2\sin x$ at which the tangent line is horizontal.

61. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

62. If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

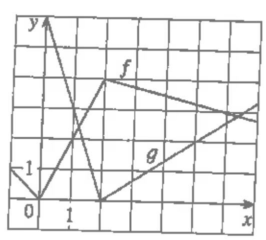
63. A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

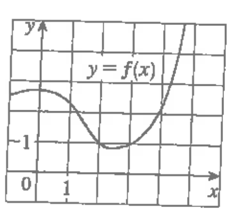
(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

64. Let f and g be the functions in Exercise 63.
(a) If $F(x) = f(f(x))$, find $F'(2)$.
(b) If $G(x) = g(g(x))$, find $G'(3)$.

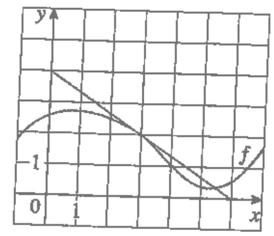
65. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if it exists. If it does not exist, explain why.
(a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$



66. If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.
(a) $h'(2)$ (b) $g'(2)$



67. If $g(x) = \sqrt{f(x)}$, where the graph of f is shown, evaluate $g'(3)$.



68. Suppose f is differentiable on $\mathbb{R}$ and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

69. Suppose f is differentiable on $\mathbb{R}$. Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

70. Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx}f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.
(a) Find $g'(0)$ and $g''(0)$ in terms of c.
(b) In terms of k, find an equation of the tangent line to the graph of h at the point where $x = 0$.

71. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

72. If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

73. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.

74. If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.

75. Show that the function $y = e^{2x}(A\cos 3x + B\sin 3x)$ satisfies the differential equation $y'' - 4y' + 13y = 0$.

76. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' - 4y' + y = 0$?

77. Find the 50th derivative of $y = \cos 2x$.

78. Find the 1000th derivative of $f(x) = xe^{-x}$.

79. The displacement of a particle on a vibrating string is given by the equation $s(t) = 10 + \frac{1}{4}\sin(10\pi t)$ where s is measured in centimeters and t in seconds. Find the velocity of the particle after t seconds.

80. If the equation of motion of a particle is given by $s = A\cos(\omega t + \delta)$, the particle is said to undergo *simple harmonic motion.*
(a) Find the velocity of the particle at time t.
(b) When is the velocity 0?

81. A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by ± 0.35. In view of these data, the brightness of Delta Cephei at time t, where t is mea-