approach 0. To do this we use our knowledge of the sine function. Because the sine of any number lies between -1 and 1, we can write

$$\boxed{4} \qquad -1 \leqslant \sin\frac{1}{x} \leqslant 1$$

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geqslant 0$ for all x and so, multiplying each side of the inequalities in $\boxed{4}$ by x^2, we get

$$-x^2 \leqslant x^2 \sin\frac{1}{x} \leqslant x^2$$

as illustrated by Figure 8. We know that

$$\lim_{x\to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x\to 0}(-x^2) = 0$$

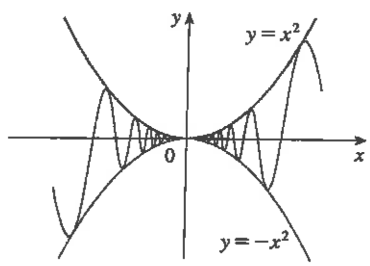


FIGURE 8
$y = x^2 \sin(1/x)$

Taking $f(x) = -x^2$, $g(x) = x^2\sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

2.3 Exercises

1. Given that

$$\lim_{x\to 2} f(x) = 4 \qquad \lim_{x\to 2} g(x) = -2 \qquad \lim_{x\to 2} h(x) = 0$$

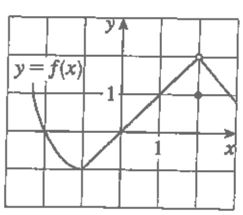
find the limits that exist. If the limit does not exist, explain why.

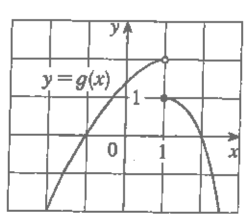
(a) $\lim_{x\to 2}[f(x) + 5g(x)]$ (b) $\lim_{x\to 2}[g(x)]^3$

(c) $\lim_{x\to 2}\sqrt{f(x)}$ (d) $\lim_{x\to 2}\dfrac{3f(x)}{g(x)}$

(e) $\lim_{x\to 2}\dfrac{g(x)}{h(x)}$ (f) $\lim_{x\to 2}\dfrac{g(x)h(x)}{f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.





(a) $\lim_{x\to 2}[f(x) + g(x)]$ (b) $\lim_{x\to 1}[f(x) + g(x)]$

(c) $\lim_{x\to 0}[f(x)g(x)]$ (d) $\lim_{x\to -1}\dfrac{f(x)}{g(x)}$

(e) $\lim_{x\to 2}[x^3 f(x)]$ (f) $\lim_{x\to 1}\sqrt{3 + f(x)}$

3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x\to -2}(3x^4 + 2x^2 - x + 1)$

4. $\lim_{x\to -1}(x^4 - 3x)(x^2 + 5x + 3)$

5. $\lim_{t\to -2}\dfrac{t^4 - 2}{2t^2 - 3t + 2}$

6. $\lim_{u\to -2}\sqrt{u^4 + 3u + 6}$

7. $\lim_{x\to 8}(1 + \sqrt[3]{x})(2 - 6x^2 + x^3)$

8. $\lim_{t\to 2}\left(\dfrac{t^2 - 2}{t^3 - 3t + 5}\right)^2$

9. $\lim_{x\to 2}\sqrt{\dfrac{2x^2 + 1}{3x - 2}}$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2}\frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2}(x + 3)$$

is correct.

Graphing calculator or computer required 1. Homework Hints available at stewartcalculus.com

11–32 Evaluate the limit, if it exists.

11. $\lim_{x\to2}\dfrac{x^2+x-6}{x-2}$

12. $\lim_{x\to-4}\dfrac{x^2+5x+4}{x^2+3x-4}$

13. $\lim_{x\to2}\dfrac{x^2-x+6}{x-2}$

14. $\lim_{x\to-1}\dfrac{x^2-4x}{x^2-3x-4}$

15. $\lim_{t\to-3}\dfrac{t^2-9}{2t^2+7t+3}$

16. $\lim_{x\to-1}\dfrac{2x^2+3x+1}{x^2-2x-3}$

17. $\lim_{h\to0}\dfrac{(-5+h)^2-25}{h}$

18. $\lim_{h\to0}\dfrac{(2+h)^3-8}{h}$

19. $\lim_{x\to-2}\dfrac{x+2}{x^3+8}$

20. $\lim_{t\to1}\dfrac{t^4-1}{t^3-1}$

21. $\lim_{h\to0}\dfrac{\sqrt{9+h}-3}{h}$

22. $\lim_{u\to2}\dfrac{\sqrt{4u+1}-3}{u-2}$

23. $\lim_{x\to-4}\dfrac{\frac{1}{4}+\frac{1}{x}}{4+x}$

24. $\lim_{x\to-1}\dfrac{x^2+2x+1}{x^4-1}$

25. $\lim_{t\to0}\dfrac{\sqrt{1+t}-\sqrt{1-t}}{t}$

26. $\lim_{t\to0}\left(\dfrac{1}{t}-\dfrac{1}{t^2+t}\right)$

27. $\lim_{x\to16}\dfrac{4-\sqrt{x}}{16x-x^2}$

28. $\lim_{h\to0}\dfrac{(3+h)^{-1}-3^{-1}}{h}$

29. $\lim_{t\to0}\left(\dfrac{1}{t\sqrt{1+t}}-\dfrac{1}{t}\right)$

30. $\lim_{x\to-4}\dfrac{\sqrt{x^2+9}-5}{x+4}$

31. $\lim_{h\to0}\dfrac{(x+h)^3-x^3}{h}$

32. $\lim_{h\to0}\dfrac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

33. (a) Estimate the value of

$$\lim_{x\to0}\frac{x}{\sqrt{1+3x}-1}$$

by graphing the function $f(x)=x/(\sqrt{1+3x}-1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

34. (a) Use a graph of

$$f(x)=\frac{\sqrt{3+x}-\sqrt{3}}{x}$$

to estimate the value of $\lim_{x\to0}f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

35. Use the Squeeze Theorem to show that $\lim_{x\to0}(x^2\cos 20\pi x)=0$. Illustrate by graphing the functions $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$, and $h(x)=x^2$ on the same screen.

36. Use the Squeeze Theorem to show that

$$\lim_{x\to0}\sqrt{x^3+x^2}\sin\frac{\pi}{x}=0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

37. If $4x-9\leqslant f(x)\leqslant x^2-4x+7$ for $x\geqslant0$, find $\lim_{x\to4}f(x)$.

38. If $2x\leqslant g(x)\leqslant x^4-x^2+2$ for all x, evaluate $\lim_{x\to1}g(x)$.

39. Prove that $\lim_{x\to0}x^4\cos\dfrac{2}{x}=0$.

40. Prove that $\lim_{x\to0^+}\sqrt{x}\,e^{\sin(\pi/x)}=0$.

41–46 Find the limit, if it exists. If the limit does not exist, explain why.

41. $\lim_{x\to3}(2x+|x-3|)$

42. $\lim_{x\to-6}\dfrac{2x+12}{|x+6|}$

43. $\lim_{x\to0.5^-}\dfrac{2x-1}{|2x^3-x^2|}$

44. $\lim_{x\to-2}\dfrac{2-|x|}{2+x}$

45. $\lim_{x\to0^-}\left(\dfrac{1}{x}-\dfrac{1}{|x|}\right)$

46. $\lim_{x\to0^+}\left(\dfrac{1}{x}-\dfrac{1}{|x|}\right)$

47. The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn}x=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } x=0\\ 1 & \text{if } x>0\end{cases}$$

(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x\to0^+}\operatorname{sgn}x$ (ii) $\lim_{x\to0^-}\operatorname{sgn}x$

(iii) $\lim_{x\to0}\operatorname{sgn}x$ (iv) $\lim_{x\to0}|\operatorname{sgn}x|$

48. Let

$$f(x)=\begin{cases}x^2+1 & \text{if } x<1\\ (x-2)^2 & \text{if } x\geqslant1\end{cases}$$

(a) Find $\lim_{x\to1^-}f(x)$ and $\lim_{x\to1^+}f(x)$.

(b) Does $\lim_{x\to1}f(x)$ exist?

(c) Sketch the graph of f.

49. Let $g(x)=\dfrac{x^2+x-6}{|x-2|}$.

(a) Find

(i) $\lim_{x\to2^+}g(x)$ (ii) $\lim_{x\to2^-}g(x)$

(b) Does $\lim_{x\to2}g(x)$ exist?

(c) Sketch the graph of g.

50. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leqslant 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.
(i) $\lim_{x\to 1^-} g(x)$ (ii) $\lim_{x\to 1} g(x)$ (iii) $g(1)$
(iv) $\lim_{x\to 2^-} g(x)$ (v) $\lim_{x\to 2^+} g(x)$ (vi) $\lim_{x\to 2} g(x)$
(b) Sketch the graph of g.

51. (a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 10, evaluate
(i) $\lim_{x\to -2^+} [\![x]\!]$ (ii) $\lim_{x\to -2} [\![x]\!]$ (iii) $\lim_{x\to -2.4} [\![x]\!]$
(b) If n is an integer, evaluate
(i) $\lim_{x\to n^-} [\![x]\!]$ (ii) $\lim_{x\to n^+} [\![x]\!]$
(c) For what values of a does $\lim_{x\to a} [\![x]\!]$ exist?

52. Let $f(x) = [\![\cos x]\!]$, $-\pi \leqslant x \leqslant \pi$.
(a) Sketch the graph of f.
(b) Evaluate each limit, if it exists.
(i) $\lim_{x\to 0} f(x)$ (ii) $\lim_{x\to (\pi/2)^-} f(x)$
(iii) $\lim_{x\to (\pi/2)^+} f(x)$ (iv) $\lim_{x\to \pi/2} f(x)$
(c) For what values of a does $\lim_{x\to a} f(x)$ exist?

53. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x\to 2} f(x)$ exists but is not equal to $f(2)$.

54. In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v\to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

55. If p is a polynomial, show that $\lim_{x\to a} p(x) = p(a)$.

56. If r is a rational function, use Exercise 55 to show that $\lim_{x\to a} r(x) = r(a)$ for every number a in the domain of r.

57. If $\lim_{x\to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim_{x\to 1} f(x)$.

58. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 5$, find the following limits.
(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 0} \dfrac{f(x)}{x}$

59. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x) = 0$.

60. Show by means of an example that $\lim_{x\to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

61. Show by means of an example that $\lim_{x\to a} [f(x)g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

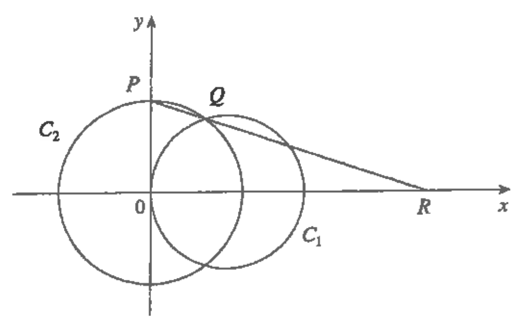
62. Evaluate $\lim_{x\to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

63. Is there a number a such that

$$\lim_{x\to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

64. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



2.4 The Precise Definition of a Limit

The intuitive definition of a limit given in Section 2.2 is inadequate for some purposes because such phrases as "x is close to 2" and "$f(x)$ gets closer and closer to L" are vague. In order to be able to prove conclusively that

$$\lim_{x\to 0}\left(x^3 + \frac{\cos 5x}{10{,}000}\right) = 0.0001 \qquad \text{or} \qquad \lim_{x\to 0}\frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.